Tutorial Sheet number 02

Exercise 1. Determine whether the following subsets of $\mathbb{R}^3$ are subspaces. Justify.

a) $W = \{(x, y, z) \mid x + y + z = 1\}$

b) $W = \{(x, y, z) \mid 2x - y = 0\}$

Exercise 2. Does the set $S = \{(1, 0, 1), (0, 1, 1)\}$ span $\mathbb{R}^3$?

Exercise 3. Determine if the vectors are linearly independent in $\mathbb{R}^4$:

$$\mathbf{v}_1 = (1, 2, 0, 1),\ \mathbf{v}_2 = (2, 1, 1, 0),\ \mathbf{v}_3 = (3, 3, 1, 1)$$

Exercise 4. Let :

$$U = \text{span}\{(1, 1, 0, 0), (0, 1, 1, 0)\}, \quad V = \text{span}\{(1, 0, 1, 0), (0, 1, 0, 1)\}$$

in $\mathbb{R}^4$.

Find bases for : a) $U \cap V$ b) $U + V$ c) Verify : $\dim(U + V) = \dim U + \dim V - \dim(U \cap V)$

Exercise 5. Let $B = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ be a basis for $\mathbb{R}^3$. Find the coordinates of $\mathbf{v} = (3, 2, 1)$ with respect to B.

Exercise 6. Is the follwoing proposition *true* or *false* ? Every subspace of $\mathbb{R}^n$ has a **unique** basis.

Exercise 7. Determine the domain of the following real functions defined by

(1) $f_1(x) = \frac{\sqrt{x+1}}{x-4}$;

(2) $f_2(x) = \frac{1}{\sqrt{|x|-x}}$;

(3) $f_3(x) = \frac{\ln(x+1)}{\lfloor x \rfloor - 2x}$;

(4) $f_4(x) = \sqrt{\frac{-3}{x^2-5x+4}}$;

(5) $f_5(x) = (f_1 \circ f_7)(x)$;

(6) $f_6(x) = \frac{f_2}{f_3}(x)$.

(7) $f_7(x) = \sin(2x - \pi)$

(8) $f_8(x) = \tan\left(\frac{x}{2}\right)$

Exercise 8. In each case, verify that T is the period of f.

(1) $f : \mathbb{R} \to \mathbb{R}, x \mapsto f(x) = 4\cos\left(2\left(x + \frac{\pi}{4}\right)\right) - 3, \quad T = \pi$;

(2) $f : \mathbb{R} \setminus \left\{\frac{5}{24\pi} + \frac{k}{4\pi} \mid k \in \mathbb{Z}\right\} \to \mathbb{R}, x \mapsto f(x) = \tan\left(4\pi^2 x - \frac{\pi}{3}\right), \quad T = \frac{1}{4\pi}$;

(3) $f : \mathbb{R} \to \mathbb{R}, x \mapsto f(x) = \lfloor x \rfloor - x, \quad T = 1$;

Exercise 9. Find the parity of the following functions of a single real variable. Justify your answer.

(1) $f : \text{dom}\, f \to \mathbb{R},\ x \mapsto f(x) = \frac{|x|}{x}$;

(2) $f : \text{dom}\, f \to \mathbb{R},\ x \mapsto f(x) = (x^2 + 4)(x - 2)(x + 2)$;

(3) $f : \text{dom}\, f \to \mathbb{R},\ x \mapsto f(x) = \sqrt{|x|}$;

(4) $f : \text{dom}\, f \to \mathbb{R},\ x \mapsto f(x) = x - \frac{1}{x}$;

(5) $f : \text{dom}\, f \to \mathbb{R},\ x \mapsto f(x) = \lfloor x \rfloor$;

Exercise 10.

1. Find the monotonicity of the following functions :
   (a) $f_1 : [0; \infty) \to \mathbb{R},\ x \mapsto \frac{1}{x^2+1}$;
   (b) $f_2 : (-1; 1) \to \mathbb{R},\ x \mapsto \sqrt{1 - x^2}$;
   (c) $f_3 : (-\infty; 0) \to \mathbb{R},\ x \mapsto e^{-x^2+x-1}$.
   (d) $f_4 : (-\infty, 8) \to \mathbb{R},\ x \mapsto \ln(-x + 8)$
2. Find the *lub* (supremum) and *glb* (infemum) of each of the following sets :

(i) $A = \left\{ \frac{1}{x^2+1} \mid x \in (0;a] \right\}$, $a > 0$; (iii) **Optional.** $C = \{\sin x \mid 0 \leq x \leq 5\pi/4\}$;

(ii) $B = \left\{ \sqrt{1-x^2} \mid x \in [-1;0] \right\}$; (iv) **Optional.** $D = \{x^2+1 \mid x \in [-5;4]\}$.

Exercise 11. Consider the hyperbolic functions defined by :

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh x = \frac{e^x - e^{-x}}{2}$$

1. Detrmine the domain of $x \mapsto \sinh x$ and $x \mapsto \cosh x$
2. Show that $x \mapsto \cosh x$ is an even function and $x \mapsto \sinh x$ is an odd function.
3. Prove the fundamental identity $\forall x \in \mathbb{R}, \quad \cosh^2 x - \sinh^2 x = 1$
4. Determine the limits :
   (a) $\lim_{x\to\infty} \cosh x$ and $\lim_{x\to-\infty} \cosh x$
   (b) $\lim_{x\to\infty} \sinh x$ and $\lim_{x\to-\infty} \sinh x$
5. Find the minimum value of $\cosh x$ and determine where it occurs.
6. Study the variations (monotonicity) of $x \mapsto \cosh x$ and $x \mapsto \sinh x$.
7. Dress the table of variations of $x \mapsto \cosh x$ and $x \mapsto \sinh x$.
8. Sketch the graphs of $y = \cosh x$ and $y = \sinh x$ on the same coordinate system.

Exercise 12. Let f be the floor function, i.e., $f : x \mapsto f(x) = \lfloor x \rfloor$.

1. Plot the graph of the function f on the interval $[-3;3]$.
2. At which points f is continuous ?
3. Among its points of discontinuity, at which points is it continuous on the left ? continuous on the right ?
4. Are the functions $g : x \mapsto g(x) = \lfloor x \rfloor + (x - \lfloor x \rfloor)^2$ and $h : x \mapsto x + |x - \lfloor x \rfloor|$ continuous.

Exercise 13. Let f be the function define on $\mathbb{R}$ by $f(x) = x^3 - \left\lfloor x - \frac{1}{3} \right\rfloor - 1$.

1. Discuss the continuity of the function f.
2. Verify that 0 is a zero of f.
3. Show that the function f admits an other zero on $(0;\infty)$.

Exercise 14. Decompose the following rational expressions into partial fractions

(a) $\frac{x}{(x+1)(x-2)}$; (c) $\frac{2x^3+x^2-x+1}{x^2-2x+1}$; (e) $\frac{x^2}{x^2+1}$;

(b) $\frac{2x}{(x^2+x+1)(x-2)}$; (d) $\frac{x+3}{(x+1)^2(x-2)}$; (f) $\frac{x^4+1}{(x+1)^2(x^2+1)}$.

Exercise 15. 1. Evaluate the following expressions, giving the answer in radians.

(a) $\arcsin\left(\sqrt{2}/2\right)$; (c) $\cos(\arctan(4))$;

(b) $\arccos\left(-\sqrt{3}/2\right)$; (d) $\cos(\arcsin(3/7))$;

2. For all $x \in \mathbb{R}$, evaluate the following expressions,

(a) $\sin(\arctan(3x))$; (b) $\cos(\arctan(4x))$.

Exercise 16. Show that :
$\forall x \in [-1;1]$,

(1) $\cos(\arcsin x) = \sqrt{1-x^2}$;

(2) $\sin(\arccos x) = \sqrt{1-x^2}$.