



Exercise 1.

1. Simplify each of the following numbers and determine the smallest set to which each belongs

$$\begin{aligned} \bullet A &= \sqrt{144 + \frac{(1+\sqrt{2})(1-\sqrt{2})}{2}} & \bullet D &= \frac{\frac{4}{3} + \frac{4}{3}}{(-\sqrt{9})^2} + 1 & \bullet F &= \sqrt{\frac{2+\sqrt{3}}{8+4\sqrt{3}}} \\ \bullet B &= \sqrt{\left(\sqrt{17}\right)^4 + \sqrt{(-2)^{12}}} & & & \bullet G &= \sqrt{17 + 12\sqrt{2}} \\ \bullet C &= \frac{4}{3} + \frac{4}{3} + \frac{1}{2} & \bullet E &= \sqrt{19 - \frac{\sqrt{8}}{\sqrt{2}}} + \sqrt{64} & & \end{aligned}$$

2. Without using a calculator, show that the numbers $L = \frac{425}{200}$ and $M = \frac{231}{350}$ are decimals.

3. Is the number $N = \frac{71}{15}$ a decimal? Justify.

4. Let α and β be real numbers where

$$\alpha = \frac{\sqrt{5}}{\sqrt{5} + \sqrt{3}}, \quad \beta = \frac{\sqrt{5}}{\sqrt{5} - \sqrt{3}}$$

- Write α and β as fractions with rational denominators.
 - Show that $\alpha + \beta$ is a natural number and $\alpha \cdot \beta$ is a rational number.
 - Express $\alpha + \beta$ in terms of $\alpha \cdot \beta$.
5. Determine the fractional form of the following rational numbers from their recurring (periodic) decimal expansions

$$A = 2.\underline{28}, \quad B = 17.\underline{5231}, \quad C = -31.\underline{172}$$

6. Among the following numbers, identify those that are prime

$$223, \quad 95578, \quad 121, \quad 111111, \quad 317, \quad 1961, \quad 633335$$

Exercise 2. Consider the numbers

$$A = \sqrt{8 + 2\sqrt{7}} - \sqrt{8 - 2\sqrt{7}}, \quad B = \sqrt{1 + 2\sqrt{3}}, \quad C = 3 - 2\sqrt{3}$$

1. What is the sign of A ?
2. Compare 3 and $2\sqrt{3}$, then deduce the sign of C .
3. Show that $A^2 - B^2 = C$, then use it to compare A and B .

Exercise 3. Let x and y be real numbers such that :

$$-4 < y < -3, \quad 1 < x < 6$$

1. Determine bounds for $x - y$ and $\frac{1}{\sqrt{3 + y^2}}$, then deduce a framing of $\frac{x - y}{\sqrt{3 + y^2}}$.
2. **Optional.** Frame $A = 2\sqrt{x + 3} + x^2 - 4$ and arrange the following expressions in ascending order (from the smallest to the largest)

$$\frac{1}{A^2}, \quad \frac{1}{A}, \quad \frac{1}{A^3}, \quad A, \quad A^2, \quad A^3$$

Exercise 4.

1. Given the system :

$$\begin{cases} \alpha + \beta = 12 \\ \alpha^2 - \beta^2 = 96 \end{cases}$$

Calculate $\alpha - \beta$.

2. Deduce the values of α and β .

3. Let x and y be two natural numbers satisfying

$$x^2 - y^2 = 401$$

4. Verify that the number 401 is prime.

5. Determine the values of x and y .

Exercise 5. Given that $U = \{-2, -1, 0, 1, 2, 3, 4, 5, 6, 7\}$, $X = \{\text{Even numbers in } U\}$, $Y = \{\text{Prime numbers in } U\}$ and $Z = \{-2, 3, 5, 6\}$,

1. Which ones are correct *i)* $2 \in U$; *ii)* $-2 \subset Z$; *iii)* $\{3\} \subset Y$; *iv)* $\emptyset \in U$; *v)* $\emptyset \subset U$; *vi)* $\{\emptyset\} \subset U$
2. Perform the following operations *i)* $X \cup Y$; *ii)* $(X \cap Y)^c$; *iii)* $X \cap (Y \setminus Z)$; *iv)* The cardinality of $X \cup Y$.

Exercise 6. Complete the following table

Form	Domain	Distance	Absolute Value
$1 \leq x \leq 7$			
	$x \in [-3, 5]$		
		$d(x; -3) \leq 4$	
			$ x - 3 < 1$

Exercise 7. Write the following expressions without using the absolute value symbol

- $A = |2 - 1| + |2 - 2\sqrt{2}| + (2 - \sqrt{2})^2$;
- $B = \frac{1}{2}|1 - 2\sqrt{2}|$;
- $C = |5 - 3\pi| + |2 - \sqrt{3}| + |2 + \sqrt{3}| - \pi$.

Exercise 8. Let x be a real variable and consider the expression

$$E(x) = |x + 3| - 2$$

1. Solve in $\mathbb{R}$

- The equations $E(x) = 0$, $E(x) + 2 = |x - 5|$
- The inequalities $E(x) \leq 0$, $E(x) + 2 < |x - 5|$

2. Let $I = [-2, 6]$ and $J = [-4, 3]$ be intervals.

- Represent them on a line with coordinate system $(O, \vec{i})$.
- Determine $I \cup J$ and $I \cap J$

Exercise 9. Determine whether each statement is *true* or *false*

1. $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$ for all $x \in \mathbb{R}$
2. $\lfloor -x \rfloor = -\lfloor x \rfloor$ for all $x \in \mathbb{R}$
3. $\lfloor x + n \rfloor = \lfloor x \rfloor + n$ for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$
4. $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$ for all $x, y \in \mathbb{R}$

Exercise 10. Solve in $\mathbb{R}$ the following equations and inequations

1. $|x + 3| = |x - 5|$
2. $|x - 3| = 7$
3. $|x - 5| = 6$
4. $|x + 3| \leq |x - 5|$
5. $|x + 4| \leq 2$
6. $|x + 3| + |x - 5| = 8$
7. $\lfloor x \rfloor = 5$
8. $\lfloor 2x \rfloor = 7$
9. $\lfloor x - 1 \rfloor = -3$
10. $\lfloor \frac{x}{2} \rfloor = 4$

Exercise 11. Write in an interval form the following sets

- i) $A = \{x \in \mathbb{R} : (x \leq 2) \text{ and } (x + 3 > 0)\}$; v) $F = \{x \in \mathbb{R} : \lfloor x \rfloor + 3 = 0\}$;
 ii) $B = \{x \in \mathbb{R} : (x - 1 \geq 0) \text{ or } (x^2 - 4x < -3)\}$; vi) $G = \{x \in \mathbb{R} : x^2 + 2|x| - 3 \leq 0\}$;
 iii) $C = \{x \in \mathbb{R} : \sqrt{x^2 - 2} \leq 2\}$.
 iv) $E = \{x \in \mathbb{R} : |x| - 1 \geq 0\}$; vii) $H = \{x \in \mathbb{R} : \sqrt{x^2 - 2} \leq 2\}$.

Exercise 12.

1. Are the following statements *true* or *false*

- (a) $\exists x \in \mathbb{N} : \forall y \in \mathbb{R}, \quad x + y > 0$; (c) $\exists x \in \mathbb{D} : \exists y \in \mathbb{Q}, \quad xy = 0$;
 (b) $\forall x \in \mathbb{R}, \exists y \in \mathbb{Z} : \quad x + y = 1$; (d) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \quad x - y = y - x$.

2. Give their negation.

Exercise 13. Prove by contraposition that

- (i) $\forall n \in \mathbb{Z}, \quad \text{if } n^3 + 5 \text{ is odd, then } n \text{ is even.}$
 (ii) **Optional.** for all real numbers a and b , if ab is irrational, then a is irrational or b is irrational.

Exercise 14. Calculate the following expressions

1. $\sum_{k=1}^4 (2k + 1)$ 4. $\sum_{n=1}^5 (-1)^n n$
 2. $\sum_{i=0}^3 i^2$ 5. $\prod_{m=2}^4 \left(1 + \frac{1}{m}\right)$
 3. $\prod_{j=1}^3 (j + 1)$

Exercise 15. Prove by mathematical induction

- a) $\forall n \in \mathbb{N}, \sum_{k=0}^n k^2 = \frac{n(n+1)(2n+1)}{6}$;
 b) **Optional.** $\forall x \in \mathbb{R}_+, \forall n \in \mathbb{N}, (1 + x)^n \geq 1 + nx$.

Exercise 16.

1. Prove by contradiction that, if x is a positive real number, then $x + \frac{49}{x} \geq 14$.
 2. **Optional.** Prove by contradiction that $\sqrt{2}$ is irrational.
 3. **Optional.** Prove by contraction that : If p is a prime number, then $\sqrt{p}$ is irrational.

Exercise 17. Optional. Expand the following expressions

1. $\sum_{i=1}^2 \sum_{j=1}^3 (i + j)$ 2. $\prod_{k=1}^2 \prod_{l=1}^2 (k \cdot l)$ 3. $\sum_{m=1}^3 \left(m \cdot \prod_{n=1}^2 (m + n) \right)$

Exercise 18. Optional. Justify the following statements

1. $\sum_{k=1}^n (a_k + b_k) = \sum_{k=1}^n a_k + \sum_{k=1}^n b_k$ 3. $\sum_{k=1}^n c \cdot a_k = c \cdot \sum_{k=1}^n a_k$
 2. $\prod_{k=1}^n (a_k \cdot b_k) = \prod_{k=1}^n a_k \cdot \prod_{k=1}^n b_k$ 4. $\prod_{k=1}^n c = c^n$