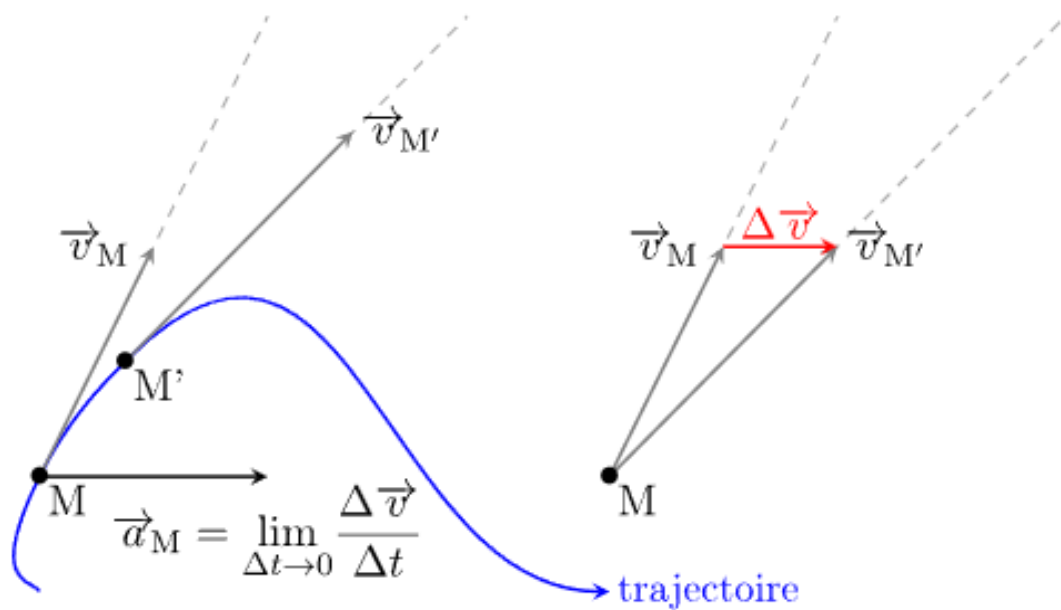


# **COURSE OF MECHANICS OF THE MATERIAL POINT**

## ***Chapter I: Kinematics of material point***

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## Glossary

| In English                            | In French                                      | In Arabic                   |
|---------------------------------------|------------------------------------------------|-----------------------------|
| Kinematics                            | La cinématique                                 | الحركات                     |
| Material point                        | Un point matériel                              | نقطة مادية                  |
| Reference system                      | Un système référentiel                         | نظام مرجعي                  |
| Velocity (speed)                      | La vitesse                                     | السرعة                      |
| Acceleration                          | L'accélération                                 | التسارع                     |
| Motion characteristics                | Caractéristiques d'un mouvement                | خصائص الحركة                |
| Position vector                       | Vecteur position                               | شعاع الموضعي                |
| Time equation/ Hourly equation        | Equation horaire                               | المعادلة الزمنية            |
| Trajectory                            | Trajectoire                                    | المسار                      |
| Trajectory equation                   | Equation de la trajectoire                     | معادلة المسار               |
| Velocity vector                       | Vecteur vitesse                                | شعاع السرعة                 |
| Acceleration vector                   | Vecteur accélération                           | شعاع التسارع                |
| Coordinate systems                    | Système de coordonnées                         | نظام الإحداثيات             |
| Cartesian coordinates                 | Coordonnées cartésiennes                       | الإحداثيات الكارتيزية       |
| Polar coordinates                     | Coordonnées polaire                            | الإحداثيات القطبية          |
| Cylindrical coordinates               | Coordonnées cylindriques                       | الإحداثيات الاسطوانية       |
| Spherical coordinates                 | Coordonnées sphériques                         | الإحداثيات الكروية          |
| Intrinsic coordinates                 | Coordonnées intrinsèques                       | الإحداثيات الجوهرية         |
| Rectilinear movement                  | Movement rectiligne                            | حركة مستقيمة                |
| Uniform rectilinear movement          | Movement rectiligne uniforme<br>MRU            | حركة مستقيمة منتظمة         |
| Uniformly varied rectilinear movement | Movement rectiligne<br>uniformement varié MRUV | حركة مستقيمة متغيرة بانتظام |
| Circular movement                     | Movement circulaire MC                         | حركة دائرية                 |
| Uniform circular movement             | Movement circulaire uniforme<br>MCU            | حركة دائرية منتظمة          |
| Uniformly varied circular movement    | Movement circulaire<br>uniformement varié MCUV | حركة دائرية متغيرة بانتظام  |

## Chapter I: Kinematics of material point

|                                   |                                          |                          |
|-----------------------------------|------------------------------------------|--------------------------|
| Sinusoidal or harmonic movement   | Mouvement sinusoïdal ou harmonique       | حركة جيبية أو توافقية    |
| A frame                           | Un referential                           | معلم او مرجع             |
| The equation of motion            | Equation de mouvement                    | معادلة الحركة            |
| A mobile                          | Un mobile                                | متحرك                    |
| Average velocity                  | La vitesse moyenne                       | السرعة المتوسطة          |
| Instantaneous velocity            | La vitesse instantanée                   | السرعة اللحظية           |
| Average acceleration              | L'accélération moyenne                   | التسارع المتوسط          |
| Instantaneous acceleration        | L'accélération instantanée               | التسارع اللحظي           |
| The orthonormal coordinate system | Un système de coordonnées orthogonal     | نظام الإحداثيات المتعامد |
| The Frenet frame                  | Le repère de Frenet / trièdre de Frenet. | معلم فرينال              |
| The moving point                  | Un point en mouvement                    | نقطة مادية في حالة حركة  |
| The normal acceleration           | L'accélération normal                    | التسارع الناطمي          |
| tangential acceleration           | L'accélération tangentielle              | التسارع المماسي          |
| Motion                            | Le Mouvement                             | الحركة                   |
| Weight                            | Le poids                                 | الوزن                    |
| Linear velocity                   | La vitesse linéaire                      | السرعة الخطية            |
| Angular velocity                  | La vitesse angulaire                     | السرعة الزاوية           |
| Linear Acceleration               | L'accélération linéaire                  | التسارع الخطي            |
| Angular Acceleration              | L'accélération angulaire                 | التسارع الزاوي           |
| Acceleration of gravity           | Accélération de pesanteur                | تسارع الجاذبية           |
| Height                            | La hauteur                               | الارتفاع                 |
| The period of a pendulum          | La période d'une pendule simple          | دور نواس بسيط            |
| The sound                         | Le son                                   | الصوت                    |
| Radius                            | Le rayon                                 | نصف قطر                  |
| The abscissa                      | L'abscisse                               | الفاصلة                  |
| Radius of curvature               | Le rayon de courbure                     | نصف قطر المسار المنحني   |
| The right triangle                | Un triangle droit                        | مثلث قائم                |
| Amplitude                         | Amplitude                                | السعة                    |
| Frequency                         | Fréquence                                | التواتر                  |
| Average speed                     | La vitesse moyenne                       | السرعة المتوسطة          |

## Chapter I: Kinematics of material point

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|                     |                        |                |
|---------------------|------------------------|----------------|
| Instantaneous speed | La vitesse instantanée | السرعة اللحظية |
|---------------------|------------------------|----------------|

## 1. Introduction

The theory of General Relativity invented by A. Einstein in 1915 is a relativistic theory of gravitation. This theory challenges the idea of an inert Euclidean space, independent of its material content. Kinematics studies the movement of a material point independently of the causes that give rise to it. It is based on a Euclidean description of space and absolute time. The material point is any material body whose dimensions are theoretically zero and practically negligible in relation to the distance it travels. The state of movement or rest of a body is two essentially relative notions: for example, a mountain is at rest in relation to the earth, but in movement in relation to an observer looking at the earth from afar, for whom the globe (with all that it contains) is in perpetual movement. In this course, we illustrate the notions of velocity and acceleration by restricting ourselves to movements in the plane.

## 2. Reference System مرجع

The concept of motion is relative. A body can be in motion with respect to one object and at rest with respect to another (relative motion), hence the necessity of choosing a reference frame.

A reference frame is a system of coordinate axes linked to an observer.

This study of motion is carried out in two forms:

- **Vectorial:** using vectors: position  $\overrightarrow{OM}$ , velocity  $\vec{v}$ , and acceleration  $\vec{a}$ .
- **Algebraic:** by defining the equation of motion along a given trajectory.

## 3. Characteristics of a movement

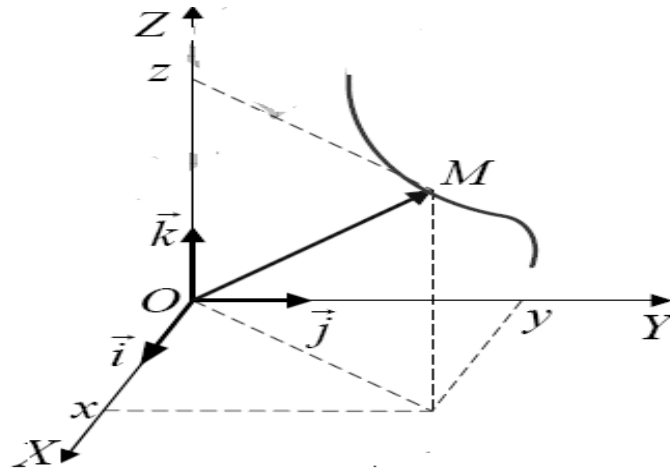
### 3.1. Vector position and time equation شعاع الموضعي و المعادلة الزمنية للحركة

We define the position of a material point M in a reference frame by the position vector  $\overrightarrow{OM}$ , where O is a fixed point and serves as the origin of the reference frame. The components of point M or the vector  $\overrightarrow{OM}$  are given in the chosen coordinate system's basis (Cartesian coordinates, polar coordinates, etc.).

The point M moves through time, and this movement is described by an equation known as the "time equation" (معادلة زمنية), translated as the "time equation."

### 3.2. Trajectory المسار

The trajectory is the geometric path of successive positions occupied by the material point over time with respect to the considered reference system.



## Example:

The position of a material point M identified by its coordinates (x, y, z) at time t in a coordinate system  $R(O, \vec{i}, \vec{j}, \vec{k})$  with a position vector:

$$\overrightarrow{OM} = (t - 1)\vec{i} + \frac{t^2}{2}\vec{j}$$

$$\overrightarrow{OM} = (t - 1)\vec{i} + \frac{t^2}{2}\vec{j} \Rightarrow \begin{cases} x = t - 1 \\ y = \frac{t^2}{2} \end{cases}$$

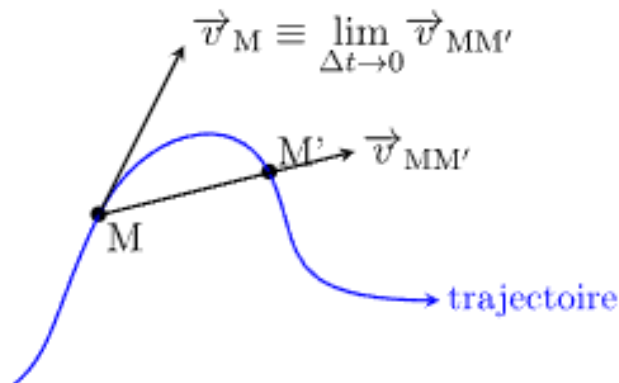
So  $t = x + 1$

The trajectory equation of the material point is

$$y = \frac{(x + 1)^2}{2}$$

## 3.3. Velocity vector شتاع السرعة

Consider a mobile that is located at position M(t) at time t, and it evolves at the point M'(t+Δt) at instant(t+Δt).



- The **average velocity** السرعة المتوسطة between the two instants  $t$  and  $t+\Delta t$  is called:

$$\overrightarrow{v_{moy}} = \frac{\overrightarrow{MM'}}{(t + \Delta t) - t} = \frac{\overrightarrow{MM'}}{\Delta t}$$

- If the time interval  $\Delta t$  is very small ( $\Delta t \rightarrow 0$ ), we then refer to it as **instantaneous velocity** السرعة اللحظية:

$$\vec{v} = \lim_{\Delta t \rightarrow 0} \overrightarrow{v_{moy}} = \lim_{\Delta t \rightarrow 0} \frac{\overrightarrow{MM'}}{\Delta t}$$

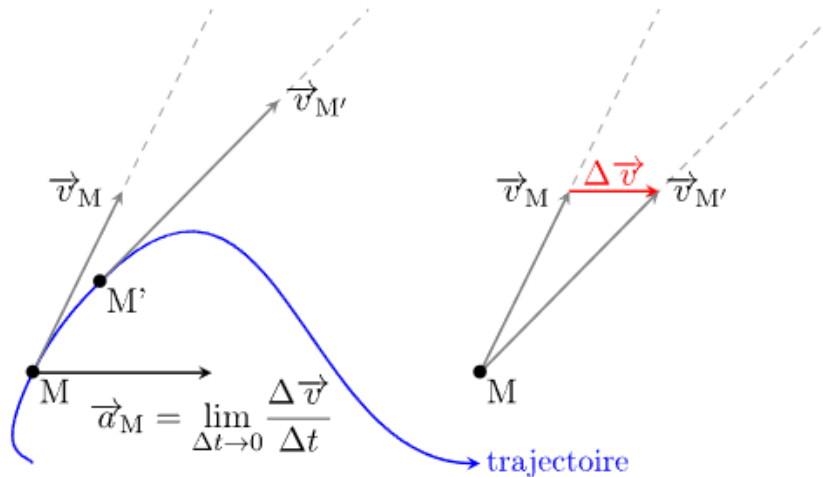
$$\overrightarrow{MM'} = \overrightarrow{MO} + \overrightarrow{OM'} = \overrightarrow{OM'} - \overrightarrow{OM} = \Delta \overrightarrow{OM}$$

So:

$$\vec{v} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \overrightarrow{OM}}{\Delta t} \Rightarrow \vec{v} = \frac{d\overrightarrow{OM}}{dt}$$

### 3.4. Acceleration vector شعاع التسارع

When velocity varies over time  $v=f(t)$ , point M is subjected to an acceleration.



- The average acceleration التسارع المتوسط is written:

$$\overrightarrow{a_{moy}} = \frac{\vec{v}(t + \Delta t) - \vec{v}(t)}{(t + \Delta t) - t} = \frac{\Delta \vec{v}(t)}{\Delta t}$$

- When the time is very small  $\Delta t \rightarrow 0$  instantaneous acceleration التسارع اللحظي is written by :

$$\vec{a} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \overrightarrow{OM}}{\Delta t}$$

$$\Rightarrow \vec{a} = \frac{d\vec{v}(t)}{dt} = \frac{d^2 \overrightarrow{OM}}{dt^2}$$

## 4. Expression of velocity and acceleration in different coordinate systems

To solve a problem in physics, we must locate the position of the moving point M in space  $\overrightarrow{OM}(t)$ .

The position must be located from a frame of reference (reference), we are required to choose the appropriate reference to use it according to the problem we want to solve

Generally, we use Cartesian, Polar, Cylindrical or Spherical coordinates

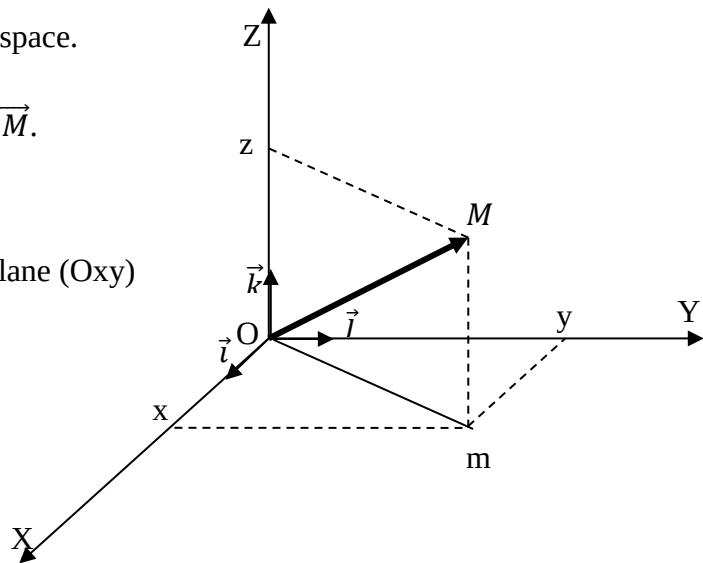
### 4.1 Cartesian coordinates

Let the frame be  $R(O,x,y,z)$  with the unit vectors  $\vec{i}, \vec{j}$  and  $\vec{k}$ . With x,y and z are the coordinates of point M which gives its position in space.

They are also the vector components  $\overrightarrow{OM}$ .

x: abscissa; y: ordinate and z: height

m is the projection of point M in the plane (Oxy)



- **Vecteur position**

$$\overrightarrow{OM} = x\vec{i} + y\vec{j} + z\vec{k}$$

The unit vectors  $\vec{i}, \vec{j}$  et  $\vec{k}$  constitute a basis linked to the axes (Ox), (Oy) and (Oz)

- **Velocity vector**

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} \Rightarrow \begin{cases} v_x = \frac{dx}{dt} \\ v_y = \frac{dy}{dt} \\ v_z = \frac{dz}{dt} \end{cases}$$

The velocity module is written:  $|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$

Note: The magnitude of the velocity, equal to  $|\vec{v}|$ , is called the speed. In S.I. units, v is expressed in (m/s) or (m.s<sup>-1</sup>).

- Acceleration vector:**

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{OM}}{dt^2} \Rightarrow \begin{cases} a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} \\ a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2} \\ a_z = \frac{dv_z}{dt} = \frac{d^2z}{dt^2} \end{cases}$$

The acceleration module is written:

$$|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

The unit of acceleration in S.I units is (m/s<sup>2</sup>) or (m.s<sup>-2</sup>).

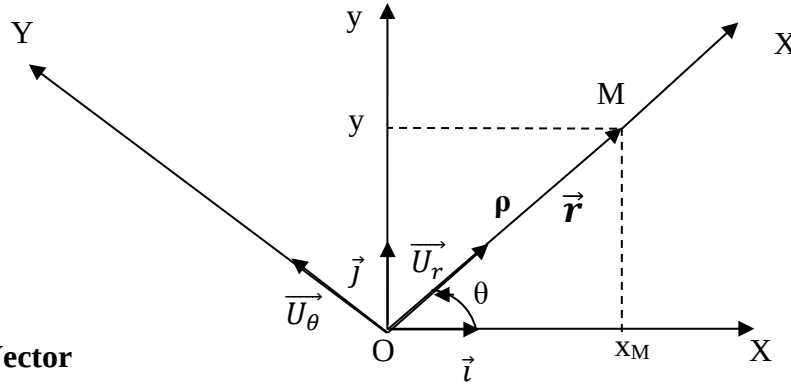
## 4.2. Polar coordinates الاحداثيات القطبية

When the motion is in a plane, it's also possible to locate the position of point M using its polar coordinates ( $\rho$ ,  $\theta$ ).

$\rho$ : polar radius  $\rho = |\vec{OM}|$  ( $0 \leq \rho \leq R$ )

$\theta$ : polar angle  $\theta = (\text{ox}, \vec{OM})$  ( $0 \leq \theta \leq 2\pi$ )

Let's consider point M moving in space, identified by its polar coordinates ( $\rho$ ,  $\theta$ ) in the orthonormal coordinate system (OXY) with unit vectors  $\vec{u}_r$ ,  $\vec{u}_\theta$ .



- Position Vector**

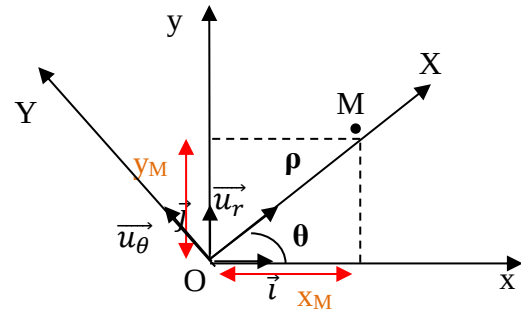
The position vector of a material point M in polar coordinates is written:  $\vec{r} = \vec{OM} = \rho \vec{u}_r$

The unit vectors  $\vec{u}_r$  is following  $\vec{OM}$  and  $\vec{u}_\theta$  is perpendicular to  $\vec{u}_r$  ( $\vec{u}_r \perp \vec{u}_\theta$ ).

### Transit relations between cartesian coordinates

### and polar coordinates

We project the point M into the plane (Oxy)



$$\begin{cases} x_M = |\overrightarrow{OM}| \cos \theta = \rho \cos \theta \\ y_M = |\overrightarrow{OM}| \sin \theta = \rho \sin \theta \end{cases}$$

$$\overrightarrow{OM} = x_M \vec{i} + y_M \vec{j} \Rightarrow \overrightarrow{OM}/cart = \rho \cos \theta \vec{i} + \rho \sin \theta \vec{j}$$

$$\overrightarrow{OM}/pol = \rho \vec{U}_r \text{ and } \overrightarrow{OM}/cart = \rho (\cos \theta \vec{i} + \sin \theta \vec{j})$$

$$\text{By identification: } \vec{u}_r = \cos \theta \vec{i} + \sin \theta \vec{j}$$

**Rule: Note: The derivative of a unit vector with respect to an angle is a unit vector perpendicular to the angle in the positive direction.**

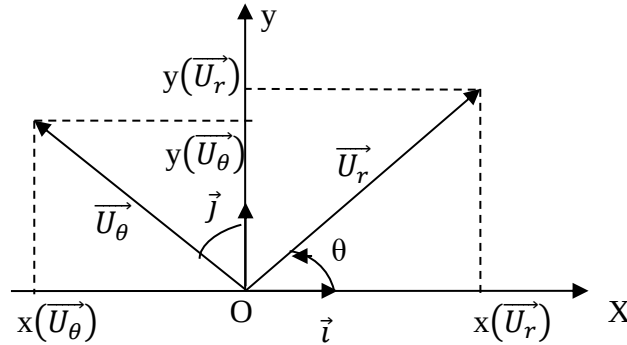
مشتقة شعاع وحدة بالنسبة إلى الزاوية هي شعاع وحدة عمودي على هذا الأخير في الاتجاه الموجب

The vector  $\vec{u}_\theta \perp \vec{u}_r$  in the direction of  $\theta$  which corresponds to the direct direction therefore

$$\vec{u}_\theta = \frac{d\vec{u}_r}{d\theta}$$

$$\text{So } \vec{u}_\theta = -\sin \theta \vec{i} + \cos \theta \vec{j}$$

By projecting the unit vectors we will have the same results



$$\vec{u}_r = x(\vec{u}_r) \vec{i} + y(\vec{u}_r) \vec{j} \Rightarrow \vec{u}_r = |\vec{u}_r| \cos \theta \vec{i} + |\vec{u}_r| \sin \theta \vec{j}$$

$$\text{with } |\vec{u}_r| = 1 \text{ since it is a unit vector therefore } \vec{u}_r = 1 \cos \theta \vec{i} + 1 \sin \theta \vec{j}$$

$$\vec{u}_\theta = x(\vec{u}_\theta) \vec{i} + y(\vec{u}_\theta) \vec{j} \Rightarrow \vec{u}_\theta = -|\vec{u}_\theta| \sin \theta \vec{i} + |\vec{u}_\theta| \cos \theta \vec{j}$$

$$\text{with } |\vec{u}_\theta| = 1 \text{ since it is a unit vector therefore } \vec{u}_\theta = -\sin \theta \vec{i} + \cos \theta \vec{j}$$

To write the unit vectors  $\vec{u}_r$  and  $\vec{u}_\theta$  as a function of  $\vec{i}$  and  $\vec{j}$  we use the passage table

|                  | $\vec{i}$      | $\vec{j}$     |
|------------------|----------------|---------------|
| $\vec{u}_r$      | $\cos \theta$  | $\sin \theta$ |
| $\vec{u}_\theta$ | $-\sin \theta$ | $\cos \theta$ |

$$\vec{i} = \cos \theta \vec{u}_r - \sin \theta \vec{u}_\theta \quad \text{and} \quad \vec{j} = \sin \theta \vec{u}_r + \cos \theta \vec{u}_\theta$$

**Example :**

Write the vector  $\vec{A} = 2x\vec{i} + y\vec{j}$

$$\begin{cases} x = \rho \cos \theta \\ y = \rho \sin \theta \end{cases} \quad \text{and} \quad \begin{cases} \vec{i} = \cos \theta \vec{u}_r - \sin \theta \vec{u}_\theta \\ \vec{j} = \sin \theta \vec{u}_r + \cos \theta \vec{u}_\theta \end{cases}$$

$$\text{So } \vec{A} = 2 \rho \cos \theta (\cos \theta \vec{u}_r - \sin \theta \vec{u}_\theta) + \rho \sin \theta (\sin \theta \vec{u}_r + \cos \theta \vec{u}_\theta)$$

$$\Rightarrow \vec{A} = 2 \rho \cos^2 \theta \vec{u}_r - 2 \rho (\cos \theta \sin \theta) \vec{u}_\theta + \rho \sin^2 \theta \vec{u}_r + \rho (\sin \theta \cos \theta) \vec{u}_\theta$$

$$\Rightarrow \vec{A} = \rho (2 \cos^2 \theta + \sin^2 \theta) \vec{u}_r - \rho (\cos \theta \sin \theta) \vec{u}_\theta$$

$$\Rightarrow \vec{A} = (\rho \cos^2 \theta + 1) \vec{u}_r - \rho (\cos \theta \sin \theta) \vec{u}_\theta$$

• **Velocity vector**

$$\vec{v} = \frac{d\vec{OM}}{dt} = \frac{d\rho}{dt} \vec{U}_r + \rho \frac{d\vec{U}_r}{dt}$$

$$\text{We have: } \frac{d\vec{U}_r}{dt} = \frac{d\vec{U}_r}{dt} \frac{d\theta}{d\theta} = \frac{d\vec{U}_r}{d\theta} \frac{d\theta}{dt}$$

$$\text{With : } \frac{d\vec{U}_r}{d\theta} = \vec{U}_\theta \quad \text{donc} \quad \frac{d\vec{U}_r}{dt} = \frac{d\theta}{dt} \vec{U}_\theta \quad \text{so} \quad \vec{v} = \frac{d\vec{OM}}{dt} = \frac{d\rho}{dt} \vec{U}_r + \rho \frac{d\theta}{dt} \vec{U}_\theta$$

$$\Rightarrow \vec{v} = \rho \cdot \vec{U}_r + \rho \theta \cdot \vec{U}_\theta \quad \text{with} \quad \rho \cdot = \frac{d\rho}{dt} \quad \text{and} \quad \theta \cdot = \frac{d\theta}{dt}$$

• **Acceleration vector**

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2 \vec{OM}}{dt^2} = \frac{d^2 \rho}{dt^2} \vec{U}_r + \frac{d\rho}{dt} \frac{d\vec{U}_r}{dt} + \frac{d\rho}{dt} \frac{d\theta}{dt} \vec{U}_\theta + \rho \frac{d^2 \theta}{dt^2} \vec{U}_\theta + \rho \frac{d\theta}{dt} \frac{d\vec{U}_\theta}{dt}$$

$$\Rightarrow \vec{a} = \frac{d^2 \rho}{dt^2} \vec{U}_r + \frac{d\rho}{dt} \frac{d\theta}{dt} \vec{U}_\theta + \frac{d\rho}{dt} \frac{d\theta}{dt} \vec{U}_\theta + \rho \frac{d^2 \theta}{dt^2} \vec{U}_\theta - \rho \left( \frac{d\theta}{dt} \right)^2 \vec{U}_r$$

$$\text{With : } \frac{d\vec{U}_r}{d\theta} = \vec{U}_\theta \quad \text{et} \quad \frac{d\vec{U}_\theta}{d\theta} = -\vec{U}_r$$

$$\text{So : } \vec{a} = \rho \cdot \vec{U}_r + 2\rho \cdot \theta \cdot \vec{U}_\theta + \rho \theta \cdot \vec{U}_\theta - \rho (\theta \cdot)^2 \vec{U}_r$$

### 4.3. Cylindrical Coordinates الاحداثيات الاسطوانية

If the spatial trajectory involves  $\rho$  and  $z$  playing a specific role in determining the position vector ( $\vec{OM}$ ); for example, the movement of air molecules in a whirlwind; it is preferable to use cylindrical coordinates  $(\rho, \theta, z)$ . With:

$\rho$ : polar radius

$\theta$ : polar angle

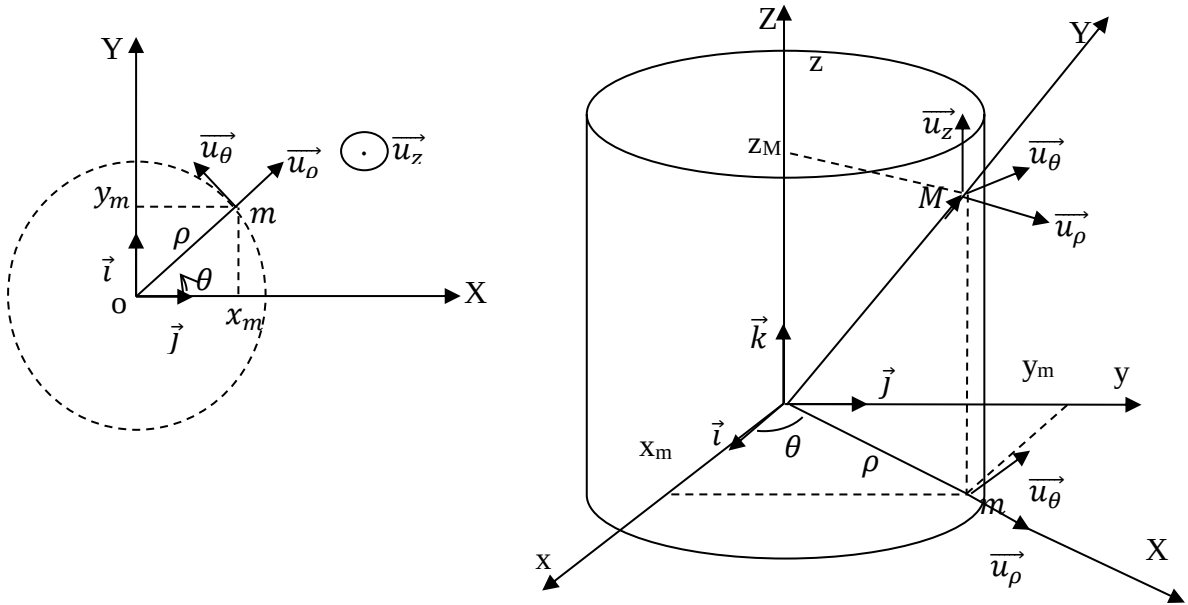
$z$ : altitude or height

$$\text{and } \begin{cases} \rho = |\overrightarrow{Om}|, & 0 < \rho < R \\ \theta = ((ox), \overrightarrow{Om}), & 0 < \theta < 2\pi \\ z = z_M, & 0 < z < H \end{cases}$$

Where  $m$  is the projection of point  $M$  onto the plane  $(Oxy)$ , and  $R$  is the radius of the cylinder, and  $H$  is the height of the cylinder.

If we add the 'z' component to polar coordinates in space, we obtain what is known as cylindrical coordinates. Consider  $R(Oxyz)$  and a point  $M$  belonging to a cylinder.

Point  $M$  is identified by three coordinates  $\rho$ ,  $\theta$  (polar coordinates), and  $z$ .



- **Position vector**

The position vector in cylindrical coordinates  $(\rho, \theta, z)$  in the orthonormal frame  $R'(O, \overrightarrow{u_\rho}, \overrightarrow{u_\theta}, \overrightarrow{u_z})$  is written:

$$\begin{cases} \vec{r} = \overrightarrow{OM} = \overrightarrow{Om} + \overrightarrow{mM} \text{ (Relation de Charles)} \\ \overrightarrow{Om} = \rho \overrightarrow{u_\rho} \text{ (Coordonnées polaires)} \\ \overrightarrow{mM} = z \overrightarrow{u_z} \text{ (hauteur du cylindre)} \end{cases} \Rightarrow \vec{r} = \overrightarrow{OM} = \rho \overrightarrow{u_\rho} + z \overrightarrow{u_z}$$

- **Unit vectors**

The unit vectors  $\overrightarrow{U}_\rho$  is following  $\overrightarrow{Om}$  ( $m$  is the projection of the point  $M$  on the plane  $(Oxy)$ ) and  $\overrightarrow{U}_\theta$  is perpendicular to  $\overrightarrow{U}_\rho$  and  $\overrightarrow{Om}$  in the direction of  $\theta$  ( $\overrightarrow{U}_\rho \perp \overrightarrow{U}_\theta$ ) and  $\overrightarrow{U}_z$  is following

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(Oz), ( $\vec{U}_z \parallel \vec{k}$ ) and it is perpendicular to the plane formed by the two other unit vectors ( $\vec{U}_\rho$  and  $\vec{U}_\theta$ ).

### Transit relations between cylindrical coordinates and Cartesian coordinates:

By projecting the point m onto the axes (Ox) and (Oy) (like polar coordinates) z is the height

$$\begin{cases} x = \rho \cos \theta \\ y = \rho \sin \theta \\ z = z \end{cases}$$

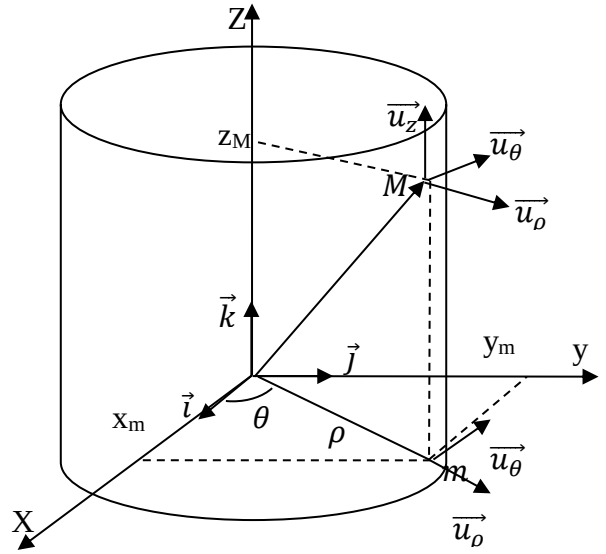
$$\vec{OM}/_{cylin} = \vec{Om} + \vec{mM} = \rho \vec{u}_\rho + z \vec{u}_z$$

$$\vec{OM}/_{cart} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{OM}/_{cart} = \rho (\cos \theta \vec{i} + \sin \theta \vec{j}) + z\vec{k}$$

**By identification**

$$\begin{cases} \vec{u}_\rho = \cos \theta \vec{i} + \sin \theta \vec{j} \\ \vec{u}_\theta = \frac{d\vec{u}_\rho}{d\theta} = -\sin \theta \vec{i} + \cos \theta \vec{j} \\ \vec{u}_z = \vec{k} \end{cases}$$



Using the passage table :

|                  | $\vec{i}$      | $\vec{j}$     | $\vec{k}$ |
|------------------|----------------|---------------|-----------|
| $\vec{u}_\rho$   | $\cos \theta$  | $\sin \theta$ | 0         |
| $\vec{u}_\theta$ | $-\sin \theta$ | $\cos \theta$ | 0         |
| $\vec{u}_z$      | 0              | 0             | 1         |

$$\vec{i} = \cos \theta \vec{u}_\rho - \sin \theta \vec{u}_\theta$$

$$\vec{j} = \sin \theta \vec{u}_\rho + \cos \theta \vec{u}_\theta \quad \text{and} \quad \vec{k} = \vec{u}_z$$

### • Velocity vector

The velocity in this case is written by:  $\vec{v} = \frac{d\vec{OM}}{dt} = \frac{d\rho}{dt} \vec{U}_r + \rho \frac{d\vec{U}_r}{dt} + \frac{dz}{dt} \vec{U}_z + z \frac{d\vec{U}_z}{dt}$

$$\frac{d\vec{U}_r}{dt} = \frac{d\vec{U}_r}{dt} \frac{d\theta}{d\theta} = \frac{d\vec{U}_r}{d\theta} \frac{d\theta}{dt}$$

With:  $\frac{d\vec{U}_r}{d\theta} = \vec{U}_\theta$  so  $\frac{d\vec{U}_r}{dt} = \frac{d\theta}{dt} \vec{U}_\theta$  and  $\frac{d\vec{U}_z}{dt} = \vec{0}$

$$\vec{v} = \frac{d\vec{OM}}{dt} = \frac{d\rho}{dt} \vec{U}_r + \rho \frac{d\theta}{dt} \vec{U}_\theta + \frac{dz}{dt} \vec{U}_z$$

$$\Rightarrow \vec{v} = \dot{\rho} \vec{U}_r + \rho \dot{\theta} \vec{U}_\theta + \dot{z} \vec{U}_z$$

With:  $\dot{\rho} = \frac{d\rho}{dt}$ ,  $\dot{\theta} = \frac{d\theta}{dt}$  and  $\dot{z} = \frac{dz}{dt}$

### • Acceleration vector

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{OM}}{dt^2} = \frac{d^2\rho}{dt^2} \vec{U}_r + \frac{d\rho}{dt} \frac{d\vec{U}_r}{dt} + \frac{d\rho}{dt} \frac{d\theta}{dt} \vec{U}_\theta + \rho \frac{d^2\theta}{dt^2} \vec{U}_\theta + \rho \frac{d\theta}{dt} \frac{d\vec{U}_\theta}{dt} + \frac{d^2z}{dt^2} \vec{U}_z + \frac{dz}{dt} \frac{d\vec{U}_z}{dt}$$

$$\Rightarrow \vec{a} = \frac{d^2\rho}{dt^2} \vec{U}_r + \frac{d\rho}{dt} \frac{d\theta}{dt} \vec{U}_\theta + \frac{d\rho}{dt} \frac{d\theta}{dt} \vec{U}_\theta + \rho \frac{d^2\theta}{dt^2} \vec{U}_\theta - \rho \left(\frac{d\theta}{dt}\right)^2 \vec{U}_r + \frac{d^2z}{dt^2} \vec{U}_z$$

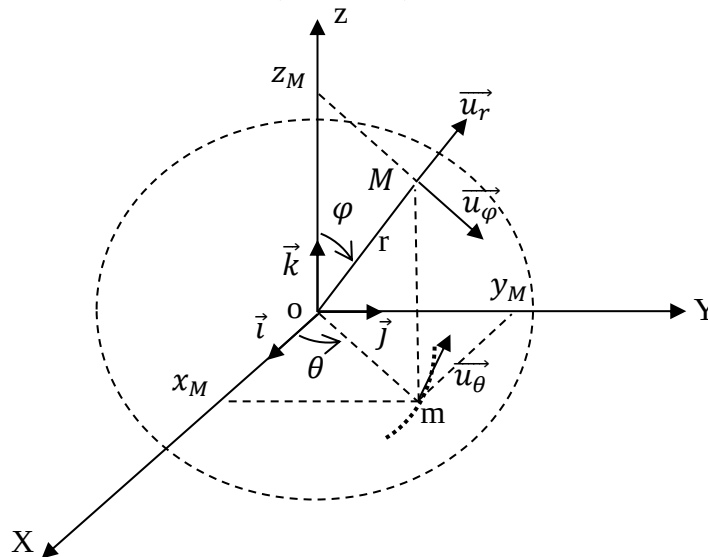
With :  $\frac{d\vec{U}_r}{d\theta} = \vec{U}_\theta$ ,  $\frac{d\vec{U}_\theta}{d\theta} = -\vec{U}_r$  and  $\frac{d\vec{U}_z}{dt} = \vec{0}$

$$\Rightarrow \vec{a} = \ddot{\rho} \vec{U}_r + 2\dot{\rho} \dot{\theta} \vec{U}_\theta + \rho \ddot{\theta} \vec{U}_\theta - \rho (\dot{\theta})^2 \vec{U}_r + \ddot{z} \vec{U}_z$$

### 4.4. Spherical coordinates *الاحداثيات الكروية*

When the point O and the distance r separating M and O play a characteristic role, the use of spherical coordinates (r,θ,φ) are the best suited in the orthonormed base ( $\vec{u}_r, \vec{u}_\theta, \vec{u}_\phi$ ) with:

$$\begin{cases} r = |\vec{OM}|, & 0 < r < R \\ \theta = ((ox), \vec{Om}) & 0 < \theta < 2\pi \\ \varphi = ((oz), \vec{OM}) & 0 < \varphi < \pi \end{cases}$$



With m is the projection of M in the plane (Oxy).

- **Position Vector :**

The position vector in spherical coordinates  $(r, \theta, \varphi)$  is written as:  $\vec{r} = \overrightarrow{OM} = r\vec{U}_r$

- **The unit vectors**

The unit vectors  $\vec{U}_r$  is following  $\overrightarrow{OM}$  and  $\vec{U}_\varphi$  is perpendicular to  $\vec{U}_r$  and  $\overrightarrow{OM}$  in the direction of  $\varphi$  ( $\vec{U}_\varphi \perp \vec{U}_r$ ) and  $\vec{U}_\theta$  is perpendicular to  $\overrightarrow{Om}$  ( $\vec{U}_\theta \perp \overrightarrow{Om}$ ).

### Transit relations between spherical coordinates and Cartesian coordinates

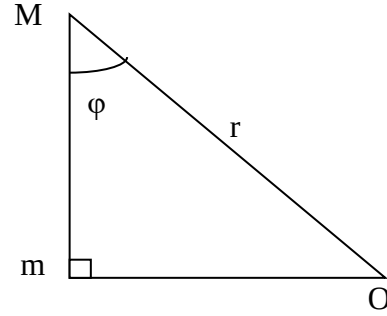
By projecting m onto the axes (Ox) and (Oy)

$$\begin{cases} x = |\overrightarrow{Om}| \cos \theta \\ y = |\overrightarrow{Om}| \sin \theta \\ z = |\overrightarrow{mM}| \end{cases}$$

By taking the right triangle (OmM):

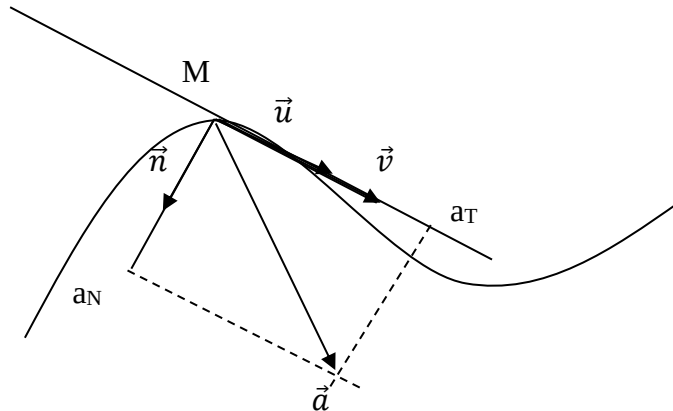
We have  $Om = r \sin \varphi$  and  $mM = r \cos \varphi$ , replacing them in passing relationships we will have:

$$\begin{cases} x = r \sin \varphi \cos \theta \\ y = r \sin \varphi \sin \theta \\ z = r \cos \varphi \end{cases}$$



### 4.5. Intrinsic coordinates (Frenet frame) احداثيات الحركة المنحنية

We used to work in a fixed frame, but in this case, we study the motion in a moving frame that travels with the moving point "M". This frame is the Frenet frame.



We study the motion in the Frenet frame:

The Frenet frame is a two-dimensional reference frame.

## Chapter I: Kinematics of material point

- $\vec{u}$  is the unit vector along the tangent to the trajectory.
- $\vec{n}$  is the unit vector normal to the trajectory and perpendicular to  $\vec{u}$ , directed towards the center of curvature.
- The position remains unchanged (the frame moves with point M).
- The velocity vector is tangent to the trajectory, and it is written as:  $\vec{v} = |\vec{v}|\vec{u}$
- The acceleration vector :

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d|\vec{v}|\vec{u}}{dt} = \frac{d|\vec{v}|}{dt}\vec{u} + |\vec{v}|\frac{d\vec{u}}{dt}$$
$$\frac{d\vec{u}}{dt} = \frac{d\vec{u}}{d\theta} \cdot \frac{d\theta}{dt} = \vec{n} \cdot \omega \text{ with } \vec{n} = \frac{d\vec{u}}{d\theta} \text{ and } \omega = \frac{d\theta}{dt}$$

The acceleration vector is written by:  $\vec{a} = a_T\vec{u} + a_N\vec{n}$

So :  $\vec{a} = \frac{d|\vec{v}|}{dt}\vec{u} + |\vec{v}| \cdot \vec{n} \cdot \omega$

(the perimeter of a circle (محيط دائرة)  $l = 2\pi R$ , for the length of a segment (طول قوس))

$x = \theta R$  ; from angular velocity to linear velocity by  $\frac{dx}{dt} = R \frac{d\theta}{dt} \Rightarrow v = R\omega$

Hence:

$\omega = \frac{v}{R}$  with R is the radius of the curvature of the trajectory.

So  $\vec{a} = \frac{d|\vec{v}|}{dt}\vec{u} + \frac{v^2}{R}\vec{n}$

The normal acceleration (التسارع الناطمي) and tangential acceleration (التسارع المماس) are written

by : 
$$\begin{cases} a_T = \frac{d|\vec{v}|}{dt} \\ a_N = \frac{v^2}{R} \end{cases}$$

$$|\vec{a}| = \sqrt{a_x^2 + a_y^2} = \sqrt{a_N^2 + a_T^2}$$

**Note :**

- $R \rightarrow \infty$  : so the trajectory is a line .
- R is constant: so the trajectory is circular.

## 5. Study of some movements

### 5.1. Rectilinear motion حركة خطية

We have linear motion if the trajectory is a straight line.

We choose a point O as the origin on the trajectory and a unit vector  $\vec{i}$  .

The position of the mobile M, as a function of time, is identified by its abscissa:

$$x(t) = \overline{OM(t)}.$$

The position vector will be:  $\overrightarrow{r(t)} = \overline{OM(t)} = x(t)\vec{i}$

### 5.1.1. Uniform rectilinear motion **حركة مستقيمة منتظمة URM**

We have uniform rectilinear motion if the trajectory is a straight line and the velocity vector is constant. This is a motion with zero acceleration  $\overline{a(t)} = \vec{0}$ .

The initial conditions to  $t=0$  ;  $x=x_0$ .

- **The velocity**

$$a = \frac{dv}{dt} = 0 \Rightarrow \int_{v_0}^v dv = \int_0^t 0 \cdot dt = cte$$

So  $v=v_0=cte$

- **The position**

$$v = \frac{dx}{dt} = v_0 \Rightarrow \int_{x_0}^x dx = \int_0^t v_0 dt = [v_0 t]_0^t = v_0 t$$

So :  $x=v_0 t+x_0$  **This is the hourly equation of the motion. URM**

### 5.1.2. Uniformly varied rectilinear motion **حركة مستقيمة متغيرة بانتظام UVRM**

One has a uniformly varied rectilinear movement if the trajectory is a straight and the acceleration is constant.

The initial conditions to  $t=0$  ;  $v=v_0$  and  $x=x_0$

- **The velocity**

$$a = \frac{dv}{dt} = a_0 \Rightarrow \int_{v_0}^v dv = \int_0^t a_0 dt = [a_0 t]_0^t$$

So  $v=a_0 t+v_0$

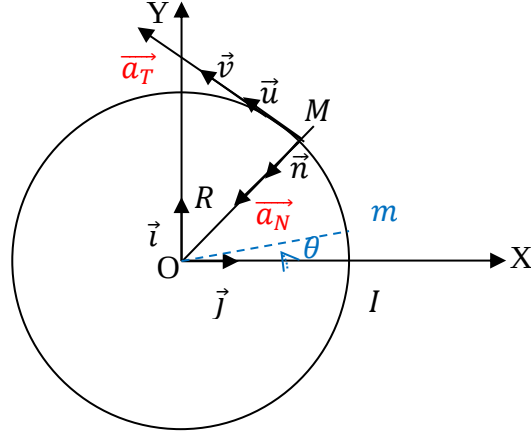
- **The position**

$$v = \frac{dx}{dt} = a_0 t + v_0 \Rightarrow \int_{x_0}^x dx = \int_0^t (a_0 t + v_0) dt = \left[ \frac{1}{2} a_0 t^2 + v_0 t \right]_0^t$$

So  $x = \frac{1}{2} a_0 t^2 + v_0 t + x_0$  **this is the hourly equation of the motion UVRM**

## 5.2. Circular motion حركية دائرية

Circular motion is plane motion with constant radius of curvature  $\rho=R$ . The trajectory of the moving object is a circle of radius  $R$ .



- The position**

The moving point travels from point I to point M, thus the trajectory forms an arc  $\widehat{IM}$ .

By considering an elementary displacement of the moving point from point I to point m, we would have a displacement in the form of an elementary arc Im.

In the right triangle OIm,  $\widehat{Im} = R \sin \theta$

In the right triangle. If  $\theta$  is so small then  $\sin \theta \approx \theta$ .

So  $\widehat{Im} = R \cdot \theta$

- The speed**

$$v = \frac{d\widehat{Im}}{dt} = R \frac{d\theta}{dt}$$

$R$  is constant, the speed is following the trajectory, so it is written  $\vec{v} = v\vec{u}$  so the vector  $\vec{u}$  would be following the tangent.

$\frac{d\theta}{dt} = \theta \cdot = \omega$  is the angular velocity السرعة الزاوية

$$v = R \frac{d\theta}{dt} = R \cdot \theta \cdot = R \cdot \omega$$

Note: The relationship between linear velocity and angular velocity is:  $\mathbf{v} = \mathbf{R}\omega$

- The acceleration**

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv}{dt}\vec{u} + v \frac{d\vec{u}}{dt}$$

$$\frac{d\vec{u}}{dt} = \frac{d\vec{u}}{d\theta} \frac{d\theta}{dt} \text{ with } \frac{d\vec{u}}{d\theta} = \vec{n}$$

(with  $(\vec{u}, \vec{n})$  the unit vectors in the Fresnet frame and  $\frac{d\theta}{dt} = \omega$ )

### 5.2.1. Uniforme circular motion حركة دائرية منتظمة

In this case the angular velocity  $\omega$  is constant and therefore the linear velocity  $v$  is also constant, then  $a_T = 0$ .

The acceleration in this case is :  $\vec{a} = \vec{a}_N = \frac{v^2}{R} \vec{n}$

### 5.2.2. Uniformly variable circular motion حركة دائرية متغيرة بانتظام

In this case the angular velocity  $\omega$  is not constant and therefore the velocity  $v$  is not constant also, then  $\vec{a} = a_T \vec{u} + a_N \vec{n}$ .

The acceleration in this case is:  $\vec{a} = \frac{dv}{dt} \vec{u} + \frac{v^2}{R} \vec{n} = R \frac{d\omega}{dt} \vec{u} + R\omega^2 \vec{n}$

### 5.3. Sinusoidal or harmonic motion حركة جيبية

The movement is called sinusoidal or harmonic if its evolution over time is written by the equation:

$$x(t) = A \sin(\omega t + \varphi)$$

A: amplitude,  $\omega$ : angular frequency, and  $\varphi$ : phase.  $\omega = \frac{2\pi}{T} = 2\pi f$

T: period and f: frequency

- **The speed**

$$v(t) = \frac{dx(t)}{dt} = A\omega \cos(\omega t + \varphi)$$

- **The acceleration**

$$a(t) = \frac{dv(t)}{dt} = \frac{d^2x(t)}{dt^2} = -A\omega^2 \sin(\omega t + \varphi) \Rightarrow a(t) = \frac{d^2x(t)}{dt^2} = -\omega^2 x(t)$$

**Note:**

**Another type of movement which is relative movement will be detailed in the next chapter.**