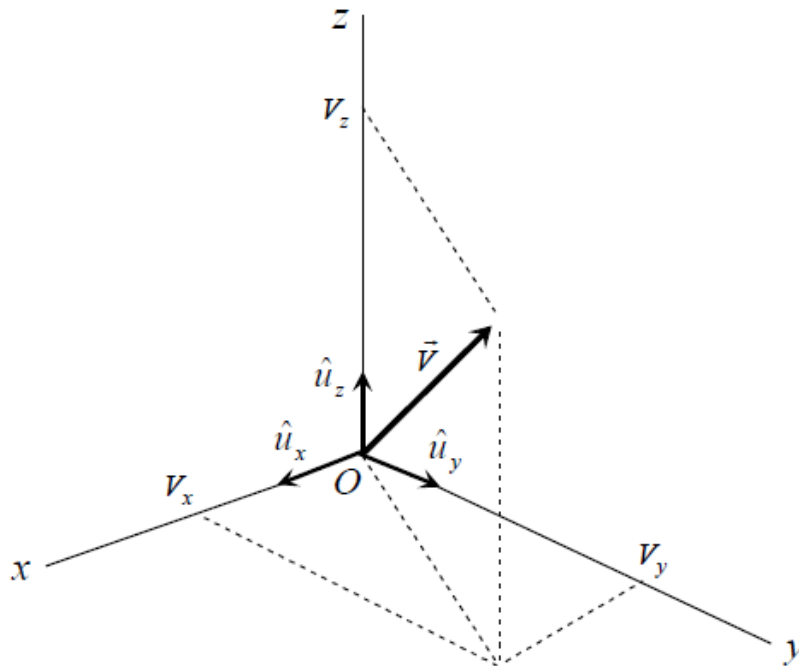


1ST YEAR LMD-MATHEMATICS
COURSE OF MECHANICS
OF THE MATERIAL POINT

Mathematical reminder on vector analysis

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1. Introduction

Vectors are fundamental mathematical entities used to represent quantities that have both magnitude and direction. Unlike scalars, which only have magnitude (e.g., distance, time, temperature), vectors provide a more comprehensive description of physical quantities by including information about their orientation or direction.

In other words, in physics, two types of quantities are used: scalar quantities and vector quantities:

- Scalar quantity **المقدار السلمي** : defined by a number (a scalar) and an appropriate unit such as: volume, mass, temperature, time ...
- Vector quantity **المقدار الشعاعي**: this is a quantity defined by a scalar, a unit and a direction such as : Displacement vector, velocity $\vec{v}$, weight $\vec{p}$, electric field ...

2. Definition

Vectors are physical or mathematical quantities carrying two properties: magnitude and direction. It is an oriented segment. Symbolically, a vector is usually represented by an arrow.



- Origin (**المبدأ**): presents the point of application "A".
- Support (**الحامل**): the straight line that carries the vector (Δ) .
- Direction (**الاتجاه**): Vectors have a specific direction or orientation in space, often indicated by angles or coordinate systems (from A to B).
- Modulus (**الطويلة**): The size or length of a vector represents its magnitude. This is typically represented by a positive numerical value gives the algebraic value of the vector $\overrightarrow{AB}$ noted.

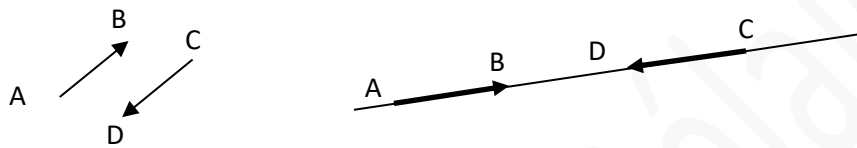
3. Vector types

- **Free vector**: the origin is not fixed.
- **Sliding vector**: the support is fixed, but the origin is not.
- **Linked vectors**: the origin is fixed.

- **Equal vectors:** if they have the same direction, the same support or parallel supports and the same modulus.



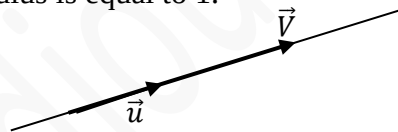
- **Opposite vector:** if they have the same support or parallel supports, the same modulus but the direction is opposite.



4. Unit Vector شعاع الوحدة

A vector is said to be unitary if its modulus is equal to 1.

We write: $|\vec{u}|=1$ and $\vec{V} = |\vec{V}| \vec{u}$



5. Algebraic measurement

Consider an axis (Δ) bearing points O and A. O is the origin, and the abscissa of point A is the algebraic measure of the vector $\overrightarrow{OA}$.

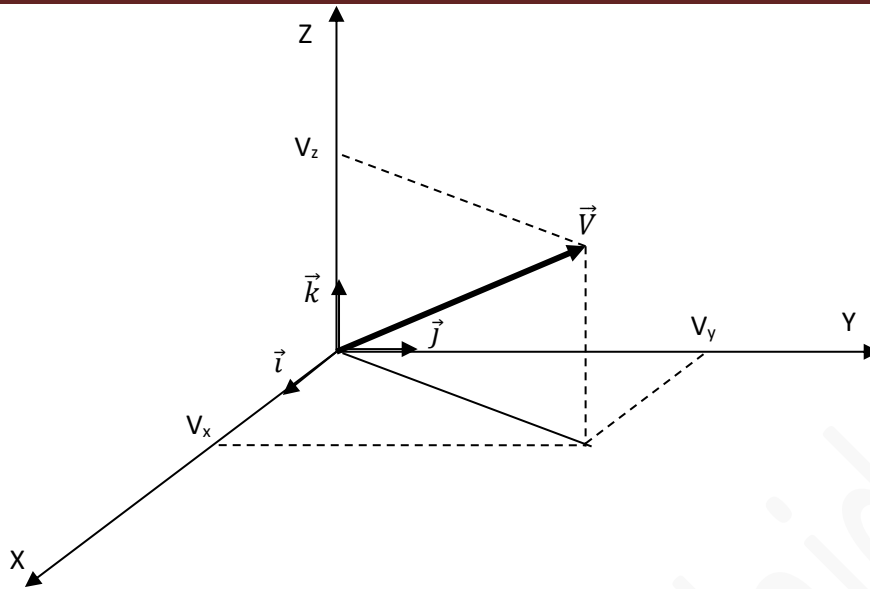


6. Components of a vector مركبات شعاع

The coordinates of a vector in space, represented in an orthonormal base frame $R(O, \vec{i}, \vec{j}, \vec{k})$ are : V_x, V_y et V_z such that:

$$\vec{V} = V_x \vec{i} + V_y \vec{j} + V_z \vec{k}$$

Where a **position vector** $\vec{V} = \overrightarrow{OM}$ is a vector used to determine the position of a point M in space, relative to a fixed reference point O which, typically, is chosen to be the origin of our coordinate system.



The modulus of the vector $\vec{V}$ is : $V = \sqrt{V_x^2 + V_y^2 + V_z^2}$

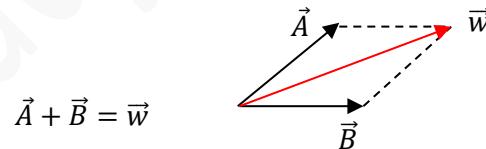
In cartesian coordinates, a vector is written as:

$$\vec{V} = x\vec{i} + y\vec{j} + z\vec{k} \Rightarrow V = \|\vec{V}\| = \sqrt{x^2 + y^2 + z^2}$$

7. Elementary operations on vectors

7.1. Vector addition

The sum of two vectors $\vec{A}$ and $\vec{B}$ is $\vec{w}$, obtained using the parallelogram:

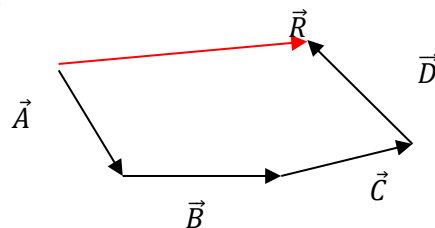


Let two vectors $\vec{A}$ and $\vec{B}$: $\vec{A} = x\vec{i} + y\vec{j} + z\vec{k}$ and $\vec{B} = x'\vec{i} + y'\vec{j} + z'\vec{k}$

$$\vec{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } \vec{B} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \text{ so } \vec{A} + \vec{B} = \vec{w} = (x + x')\vec{i} + (y + y')\vec{j} + (z + z')\vec{k}$$

Note :

- For several vectors: $\vec{A} + \vec{B} + \vec{C} + \vec{D} = \vec{R}$



2. Properties :

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}, \quad (\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C}), \quad \vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$

3. Charles relationship:

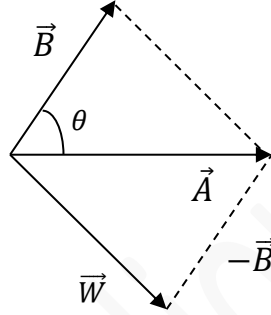
$$\text{Or the three points: A, B and C, we have: } \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

7.2. Subtracting two vectors

This is an anticommutative operation such that: $\vec{W} = \vec{A} - \vec{B} = \vec{A} + (-\vec{B})$

Let two vectors: $\vec{A}$ and $\vec{B}$, $\vec{A} = x\vec{i} + y\vec{j} + z\vec{k}$ et $\vec{B} = x'\vec{i} + y'\vec{j} + z'\vec{k}$

$$\vec{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } \vec{B} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \text{ so } \vec{A} - \vec{B} = \vec{W} = (x - x')\vec{i} + (y - y')\vec{j} + (z - z')\vec{k}$$



7.3. Product of a vector and a scalar

The product of a vector $\vec{v}$ by a scalar α is the vector $\alpha\vec{v}$, this vector has the same support as $\vec{v}$.

The two vectors ($\vec{v}$ and $\alpha\vec{v}$) have the same direction if $\alpha > 0$ and they are opposite supports if $\alpha < 0$.

$$\alpha\vec{v} = \alpha \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha x \vec{i} + \alpha y \vec{j} + \alpha z \vec{k}$$

Notes: $|\alpha\vec{v}| = |\alpha||\vec{v}|$, $\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v}$ and $(\alpha + \beta)\vec{u} = \alpha\vec{u} + \beta\vec{u}$

8. Products

8.1. Scalar product الجداء السلمي

Given two vectors $\vec{A}$ and $\vec{B}$ making an angle θ between them, the scalar product $\vec{A} \cdot \vec{B} = m$ with m is a scalar such that:

$$\vec{A} \cdot \vec{B} = m = |\vec{A}| \cdot |\vec{B}| \cos(\vec{A}, \vec{B})$$

With : $(\vec{A}, \vec{B}) = \theta$

Note : The properties of the scalar product are:

- The scalar product is commutative $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
- The scalar product isn't associative $\vec{V}_1 \cdot (\vec{V}_2 \cdot \vec{V}_3)$, doesn't exist, because the result would be a vector.
- $\vec{A} \cdot \vec{B} = 0$ when both vectors are perpendicular ($\vec{A} \perp \vec{B}$).
- If $\vec{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\vec{B} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$ so $\vec{A} \cdot \vec{B} = x \cdot x' + y \cdot y' + z \cdot z'$

8.2. Vector product الجداء الشعاعي

The vector product of two vectors $\vec{A}$ and $\vec{B}$ is a vector $\vec{C}$ and is written as:

$$\vec{C} = \vec{A} \wedge \vec{B}$$

To calculate the vector product of two vectors $\vec{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\vec{B} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$ we have :

$$\vec{A} \wedge \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ x' & y' & z' \end{vmatrix} = \vec{i} \begin{vmatrix} y & z \\ y' & z' \end{vmatrix} - \vec{j} \begin{vmatrix} x & z \\ x' & z' \end{vmatrix} + \vec{k} \begin{vmatrix} x & y \\ x' & y' \end{vmatrix} = \vec{C}$$

$$\vec{A} \wedge \vec{B} = \vec{i}(yz' - zy') - \vec{j}(xz' - zx') + \vec{k}(xy' - yx') = \vec{C}$$

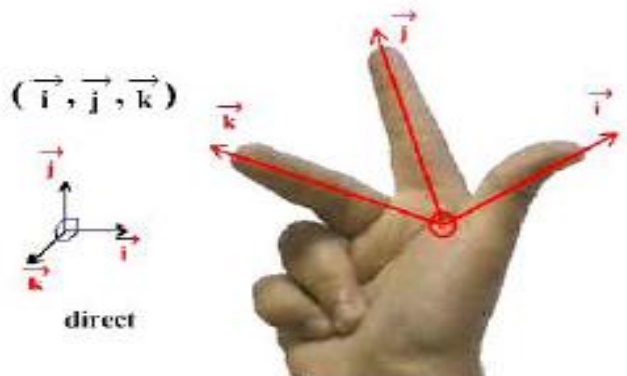
So the modulus of the vector product can be given by another method such as:

$$W = \sqrt{(yz' - zy')^2 + (xz' - zx')^2 + (xy' - yx')^2}$$

Characteristics of vector $\vec{C}$:

The support : $\vec{C}$ is perpendicular to the plane formed by the two vectors $\vec{A}$ and $\vec{B}$.

The direction: the three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ form a direct trihedron. The direction is given by the rule of the three fingers of the right hand.



The modulus :

$$|\vec{C}| = |\vec{A}| \cdot |\vec{B}| \sin(\vec{A}, \vec{B})$$

The modulus of the vector product corresponds to the area (the surface مساحة) of the parallelogram (متوازي الاضلاع) formed by the two vectors $\vec{A}$ and $\vec{B}$.

Example:

In an orthonormal Cartesian coordinate base $(\vec{i}, \vec{j}, \vec{k})$:

$$\vec{i} \wedge \vec{j} = \vec{k}, \vec{j} \wedge \vec{k} = \vec{i} \text{ et } \vec{k} \wedge \vec{i} = \vec{j}. \text{ On the other hand } \vec{i} \wedge \vec{k} = -\vec{j}$$

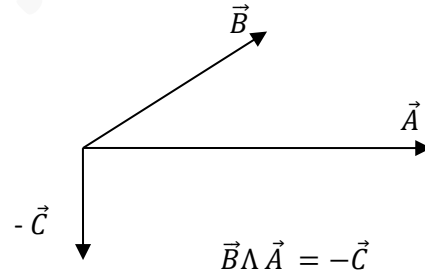
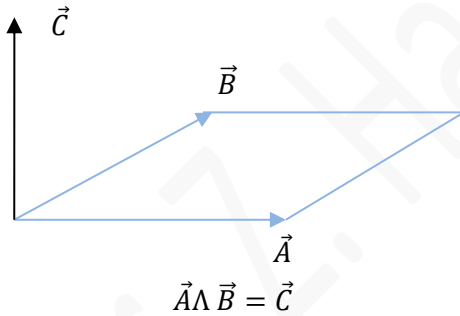
Notes : The properties of the vector product are:

- The vector product is not commutative (Anticommutative).
- Not associative : $\vec{V}_1 \wedge (\vec{V}_2 \wedge \vec{V}_3) \neq (\vec{V}_1 \wedge \vec{V}_2) \wedge \vec{V}_3$.
- Distributive with respect to vector sum: $\vec{A} \wedge (\vec{B}_1 + \vec{B}_2) = \vec{A} \wedge \vec{B}_1 + \vec{A} \wedge \vec{B}_2$

But :

$$\vec{V}_1 \wedge (\vec{V}_2 + \vec{V}_3) \neq (\vec{V}_1 \wedge \vec{V}_2) + (\vec{V}_1 \wedge \vec{V}_3)$$

- $\vec{A} \wedge \vec{B} = -\vec{B} \wedge \vec{A}$ car $\sin(\vec{A}, \vec{B}) = -\sin(\vec{B}, \vec{A})$



- $\vec{A} \wedge \vec{B} = \vec{0}$ when the two vectors are parallel ($\vec{A} \parallel \vec{B}$)

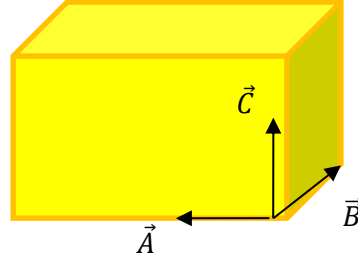
8.3. Mixed product

The mixed product of three vectors is $\vec{A}, \vec{B}$ and $\vec{C}$ a scalar quantity m such that:

$$m = (\vec{A} \wedge \vec{B}) \cdot \vec{C}$$

Mathematical reminder on vector analysis

Where $\mathbf{m}$ represents the volume of the parallelepiped (حجم متوازي المستطيلات) constructed by the three vectors :



Note: The mixed product is commutative, $(\vec{A} \wedge \vec{B}) \cdot \vec{C} = \vec{A} \cdot (\vec{B} \wedge \vec{C}) = (\vec{C} \wedge \vec{A}) \cdot \vec{B}$

9. Derivative of a vector

Let the vector $\vec{A} = x\vec{i} + y\vec{j} + z\vec{k}$ which varies with time:

Its first derivative in relation to time is:

$$\vec{A}' = \frac{d\vec{A}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

The second derivative is:

$$\vec{A}'' = \frac{d^2\vec{A}}{dt^2} = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k}$$

Note :

- Derivative of a scalar product $(\vec{A} \cdot \vec{B})' = \vec{A}' \cdot \vec{B} + \vec{A} \cdot \vec{B}'$
- If $\vec{B}$ is constant $(\vec{A} \cdot \vec{B})' = \vec{A}' \cdot \vec{B}$
- $(\vec{A}^2)' = 0$ because $(\vec{A}^2)' = 2\vec{A}' \cdot \vec{A} = 0$
- The derivative vector is perpendicular to the vector.
- A vector is written as $\vec{A} = |\vec{A}|\vec{u} = A\vec{u}$, if $\vec{u}$ is a variable vector, then $\vec{A}' = A'\vec{u} + A\vec{u}'$.

Example: The position vector on Cartesian Coordinate is written as:

$$\vec{A} = x\vec{i} + y\vec{j} + z\vec{k}$$

The velocity vector in Cartesian Coordinates is written as:

$$\vec{V} = \frac{d\vec{OM}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

The acceleration vector in Cartesian Coordinates is written as:

$$\vec{a} = \frac{d^2\vec{OM}}{dt^2} = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k}$$

Exercise

We give the three vectors $\vec{V}_1(1, 1, 0)$, $\vec{V}_2(0, 1, 0)$ and $\vec{V}_3(0, 0, 2)$.

1. Calculate norms $\|\vec{V}_1\|$, $\|\vec{V}_2\|$ and $\|\vec{V}_3\|$, deduce the unit vectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ respectively from $\vec{V}_1$, $\vec{V}_2$ and $\vec{V}_3$.
2. Calculate $\cos(\widehat{\vec{v}_1, \vec{v}_2})$, knowing that the corresponding angle is between 0 and π .
3. Calculate the mixed product $\vec{v}_1 \cdot (\vec{v}_2 \wedge \vec{v}_3)$. What does this product represent?

References

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3. M.D. Greenberg, *Advanced Engineering Mathematics*, 2nd Edition (Prentice-Hall, 1998).