University of Tlemcen
Faculty of Sciences
Department of Mathematics

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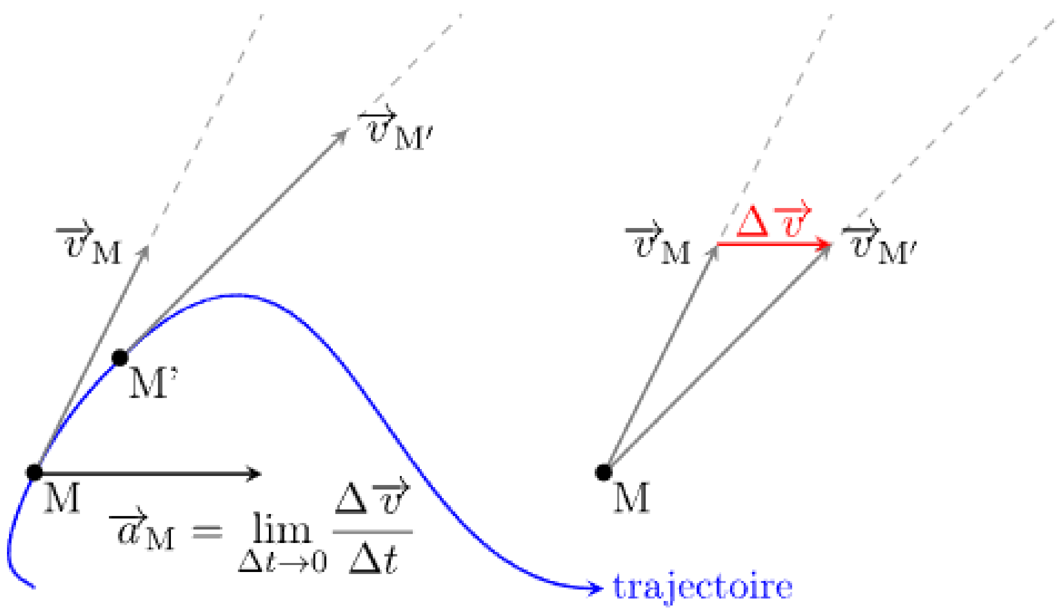
COURSE OF MECHANICS

OF THE MATERIAL POINT

Chapter III: Kinematics of material point



Glossary

In English	In French	In Arabic
Kinematics	La cinématique	الحركيات
Material point	Un point matériel	نقطة مادية
Reference system	Un système référentiel	نظام مرجعي
Velocity (speed)	La vitesse	السرعة
Acceleration	L'accélération	التسارع
Motion characteristics	Caractéristiques d'un mouvement	خصائص الحركة
Position vector	Vecteur position	شعاع الموضعي
Time equation/ Hourly equation	Equation horaire	المعادلة الزمنية
Trajectory	Trajectoire	المسار
Trajectory equation	Equation de la trajetoire	معادلة المسار
Velocity vector	Vecteur vitesse	شعاع السرعة
Acceleration vector	Vecteur accélération	شعاع التسارع
Coordinate systems	Système de coordonnée	نظام الاحداثيات
Cartesian coordinates	Coordonnées cartésiennes	الاحداثيات الكارتيزية
Polar coordinates	Coordonnées polaire	الاحداثيات القطبية
Cylindrical coordinates	Coordonnées cylindriques	الاحداثيات الاسطوانية
Spherical coordinates	Coordonnées sphériques	الاحداثيات الكروية
Intrinsic coordinates	Coordonnées intrinsèques	الإحداثيات الجوهرية
Rectilinear movement	Movement réctiligne	حركة مستقيمة
Uniform rectilinear movement	Movement réctiligne uniforme MRU	حركة مستقيمة منتظمة
Uniformly varied rectilinear movement	Movement réctiligne uniformement varié MRUV	حركة مستقيمة متغيرة بانتظام
Circular movement	Movement circulaire MC	حركة دائرية
Uniform circular movement	Movement circulaire uniforme MCU	حركة دائرية منتظمة
Uniformly varied circular movement	Movement circulaire uniformement varié MCUV	حركة دائرية متغيرة بانتظام

Sinusoidal or harmonic movement	Mouvement sinusoïdal ou harmonique	حركة جيبية أو توافقية
A frame	Un referential	معلم او مرجع
The equation of motion	Equation de mouvement	معادلة الحركة
A mobile	Un mobile	متحرك
Average velocity	La vitesse moyenne	السرعة المتوسطة
Instantaneous velocity	La vitesse instantanée	السرعة اللحظية
Average acceleration	L'accélération moyenne	التسارع المتوسط
Instantaneous acceleration	L'accélération instantanée	التسارع اللحظي
The orthonormal coordinate system	Un système de coordonnées orthogonal	نظام الإحداثيات المتعامد
The Frenet frame	Le repère de Frenet / trièdre de Frenet.	معلم فرينال
The moving point	Un point en movement	نقطة مادية في حالة حركة
The normal acceleration	L'accélération normal	التسارع الناظمي
tangential acceleration	L'accélération tangentielle	التسارع المماسي
Motion	Le Mouvement	الحركة
Weight	Le poids	الوزن
Linear velocity	La vitesse linéaire	السرعة الخطية
Angular velocity	La vitesse angulaire	السرعة الزاوية
Linear Acceleration	L'accélération linéaire	التسارع الخطي
Angular Acceleration	L'accélération angulaire	التسارع الزاوي
Acceleration of gravity	Accélération de pesanteur	تسارع الجاذبية
Height	La hauteur	الارتفاع
The period of a pendulum	La pèriode d'une pendule simple	دور نواس بسيط
The sound	Le son	الصوت
Radius	Le rayon	نصف قطر
The abscissa	L'abscisse	الفاصلة
Radius of curvature	Le rayon de courbure	نصف قطر المسار المنحني
The right triangle	Un triangle droit	مثلث قائم
Amplitude	Amplitude	السعة
Frequency	Fréquence	التواتر
Average speed	La vitesse moyenne	السرعة المتوسطة

Instantaneous speed	La vitesse instantanée	السرعة اللحضية

1. Introduction

The theory of General Relativity invented by A. Einstein in 1915 is a relativistic theory of gravitation. This theory challenges the idea of an inert Euclidean space, independent of its material content. Kinematics studies the movement of a material point independently of the causes that give rise to it. It is based on a Euclidean description of space and absolute time. The material point is any material body whose dimensions are theoretically zero and practically negligible in relation to the distance it travels. The state of movement or rest of a body is two essentially relative notions: for example, a mountain is at rest in relation to the earth, but in movement in relation to an observer looking at the earth from afar, for whom the globe (with all that it contains) is in perpetual movement. In this course, we illustrate the notions of velocity and acceleration by restricting ourselves to movements in the plane.

2. Reference Systemمرجع

The concept of motion is relative. A body can be in motion with respect to one object and at rest with respect to another (relative motion), hence the necessity of choosing a reference frame. A reference frame is a system of coordinate axes linked to an observer.
This study of motion is carried out in two forms:

- **Vectorial:** using vectors: position $\overrightarrow{OM}$, velocity $\vec{v}$, and acceleration $\vec{a}$.
- **Algebraic:** by defining the equation of motion along a given trajectory.

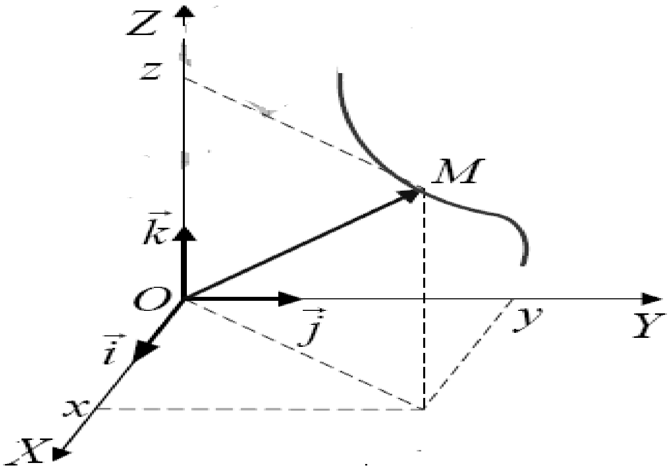
3. Characteristics of a movement

3.1. Vector position and time equationشعاع الموضعي و المعادلة الزمنية للحركة

We define the position of a material point M in a reference frame by the position vector $\overrightarrow{OM}$, where O is a fixed point and serves as the origin of the reference frame. The components of point M or the vector $\overrightarrow{OM}$ are given in the chosen coordinate system's basis (Cartesian coordinates, polar coordinates, etc.).
The point M moves through time, and this movement is described by an equation known as the "time equation" (معادلة زمنية), translated as the "time equation."

3.2.Trajectory المسار

The trajectory is the geometric path of successive positions occupied by the material point over time with respect to the considered reference system.



Example:

The position of a material point M identified by its coordinates (x, y, z) at time t in a coordinate system R($O,\vec{\imath}, \vec{\jmath},\vec{k}$) with a position vector:

$$\overrightarrow{OM} = (t-1)\vec{\imath} + \frac{t^2}{2}\vec{\jmath}$$

$$\overrightarrow{OM} = (t-1)\vec{\imath} + \frac{t^2}{2}\vec{\jmath} \quad \Rightarrow \begin{cases} x = t-1 \\ y = \dfrac{t^2}{2} \end{cases}$$

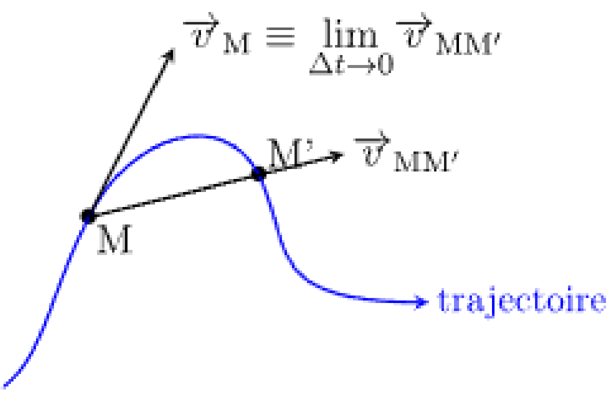
So $t=x+1$

The trajectory equation of the material point is

$$\boldsymbol{y = \frac{(x+1)^2}{2}}$$

3.3. Velocity vector شعاع السرعة

Consider a mobile that is located at position M(t) at time t, and it evolves at the point M'(t+Δt) at instant(t+Δt).



- The **average velocity** السرعة المتوسطة between the two instants t and t+Δt is called:

$$\overrightarrow{v_{moy}} = \frac{\overrightarrow{MM'}}{(t+\Delta t)-t} = \frac{\overrightarrow{MM'}}{\Delta t}$$

- If the time interval Δt is very small (Δt→0), we then refer to it as **instantaneous velocity** السرعة اللحضية**:**

$$\vec{v} = \lim_{\Delta t \to 0} \overrightarrow{v_{moy}} = \lim_{\Delta t \to 0} \frac{\overrightarrow{MM'}}{\Delta t}$$

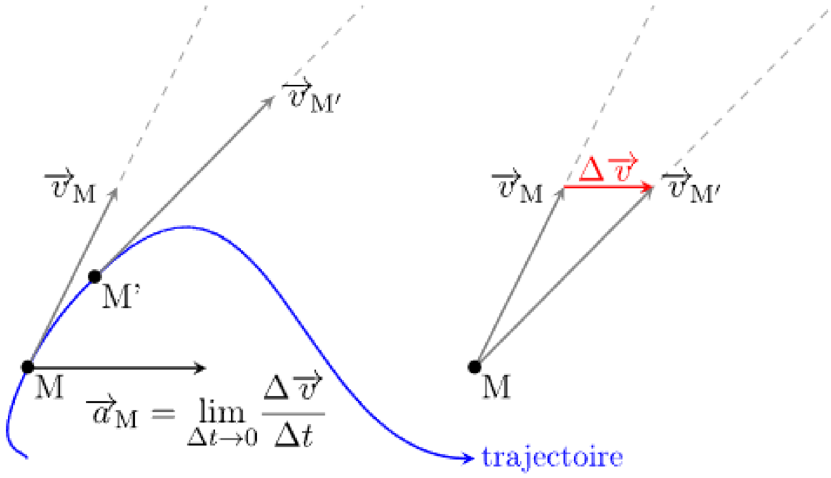
$$\overrightarrow{MM'} = \overrightarrow{MO} + \overrightarrow{OM'} = \overrightarrow{OM'} - \overrightarrow{OM} = \Delta\overrightarrow{OM}$$

So:

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\overrightarrow{OM}}{\Delta t} \Rightarrow \boldsymbol{\vec{v}} = \frac{\boldsymbol{d\overrightarrow{OM}}}{\boldsymbol{dt}}$$

3.4. Acceleration vector شعاع التسارع

When velocity varies over time v=f(t), point M is subjected to an acceleration.



- The average acceleration*التسارع المتوسط* is written:

$$\overrightarrow{a_{moy}} = \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{(t+\Delta t)-t} = \frac{\Delta\vec{v}(t)}{\Delta t}$$

- When the time is very small $\Delta t \to 0$ instantaneous acceleration *التسارع اللحضي* is written by :

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\overrightarrow{OM}}{\Delta t}$$

$$\Rightarrow \boldsymbol{\vec{a}} = \frac{\boldsymbol{d\vec{v}(t)}}{\boldsymbol{dt}} = \frac{\boldsymbol{d^2\overrightarrow{OM}}}{\boldsymbol{dt^2}}$$

4. Expression of velocity and acceleration in different coordinate systems

To solve a problem in physics, we must locate the position of the moving point M in space $\overrightarrow{OM}(t)$.

The position must be located from a frame of reference (reference), we are required to choose the appropriate reference to use it according to the problem we want to solve

Generally, we use Cartesian, Polar, Cylindrical or Spherical coordinates

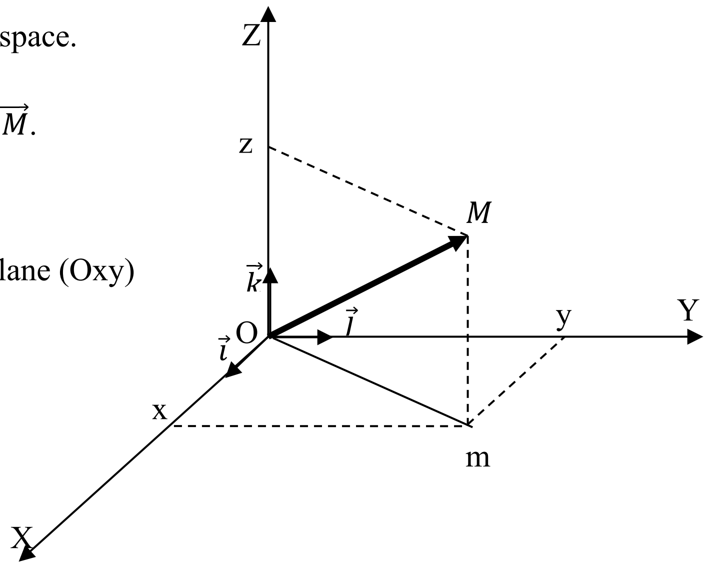
4.1 Cartesian coordinates

Let the frame be R(O,x,y,z) with the unit vectors $\vec{i}, \vec{j}$ *and* $\vec{k}$. With x,y and z are the coordinates of point M which gives its position in space.

They are also the vector components $\overrightarrow{OM}$.

x: abscissa; y: ordinate and z: height

m is the projection of point M in the plane (Oxy)



- **Vecteur position**

$\overrightarrow{OM} = x\vec{i} + y\vec{j} + z\vec{k}$

The unit vectors $\vec{i}, \vec{j}$ *et* $\vec{k}$ constitute a basis linked to the axes (Ox), (Oy) and (Oz)

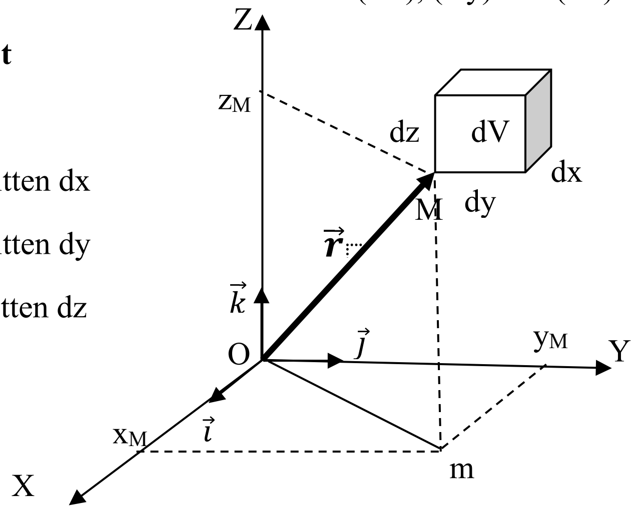
- **Elementary displacement**

The elementary displacement dl:

Next (Ox) the displacement is written dx

Next (Oy) the displacement is written dy

Next (Oz) the displacement is written dz



By fixing y and z, M moves along $\vec{i}$, the elementary displacement is then written $\overrightarrow{dl_1} = dx\vec{i}$.

By fixing x and z, M moves along $\vec{j}$, the elementary movement is then written $\overrightarrow{dl_2} = dy\vec{j}$.

By fixing x and y, M moves along $\vec{k}$, the elementary displacement is then written $\overrightarrow{dl_3} = dz\vec{k}$.

The total displacement of point M is:

$$\overrightarrow{dl} = \overrightarrow{dl_1} + \overrightarrow{dl_2} + \overrightarrow{dl_3} = dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k}$$

Or mathematically :

$$\overrightarrow{OM} = x\vec{\imath} + y\vec{\jmath} + z\vec{k} \Rightarrow d\overrightarrow{OM} = dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k}$$

The elementary volume $dV=dl_1.dl_2.dl_3=dx.dy.dz$

- **Velocity vector**

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} + \frac{dz}{dt}\vec{k} \Rightarrow \begin{cases} v_x = \dfrac{dx}{dt} \\ v_y = \dfrac{dy}{dt} \\ v_z = \dfrac{dz}{dt} \end{cases}$$

The velocity module is written: $|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$

Note: The magnitude of the velocity, equal to |v|, is called the speed. In S.I. units, v is expressed in (m/s) or ($m.s^{-1}$).

- **Acceleration vector**:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\overrightarrow{OM}}{dt^2} \Rightarrow \begin{cases} a_x = \dfrac{dv_x}{dt} = \dfrac{d^2x}{dt^2} \\ a_y = \dfrac{dv_y}{dt} = \dfrac{d^2y}{dt^2} \\ a_z = \dfrac{dv_z}{dt} = \dfrac{d^2z}{dt^2} \end{cases}$$

The acceleration module is written:

$$|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

The unit of acceleration in S.I units is (m/s^2) or ($m.s^{-2}$).

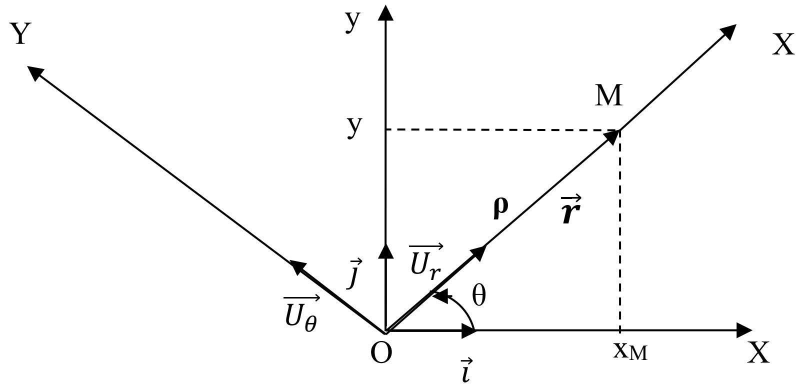
4.2. Polar coordinates الاحداثيات القطبية

When the motion is in a plane, it's also possible to locate the position of point M using its polar coordinates (ρ, θ).

ρ: polar radius $\rho = |\overrightarrow{OM}|$ $(0 \leq \rho \leq R)$

θ: polar angle $\theta = (ox, \overrightarrow{OM})$ $(0 \leq \theta \leq 2\pi)$

Let's consider point M moving in space, identified by its polar coordinates (ρ, θ) in the orthonormal coordinate system (OXY) with unit vectors $\vec{u_r}$, $\vec{u_\theta}$.



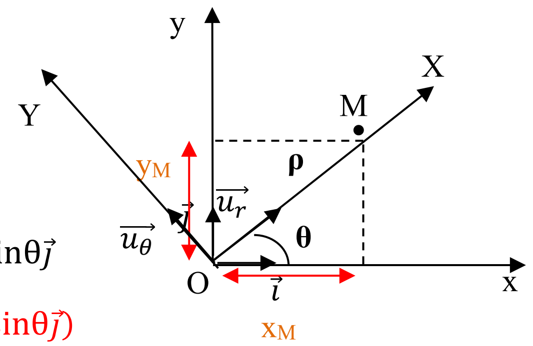
- **Position Vector**

The position vector of a material point M in polar coordinates is written: $\vec{r} = \overrightarrow{OM} = \rho\vec{U}_r$

The unit vectors $\vec{U}_r$ is following $\overrightarrow{OM}$ and $\vec{U}_\theta$ is perpendicular to $\vec{U}_r$ ($\vec{U}_r \perp \vec{U}_\theta$).

Transit relations between cartesian coordinates and polar coordinates

We project the point M into the plane (Oxy)



$$\begin{cases} x_M = |\overrightarrow{OM}|\cos\theta = \rho\cos\theta \\ y_M = |\overrightarrow{OM}|\sin\theta = \rho\sin\theta \end{cases}$$

$\overrightarrow{OM} = x_M\vec{\imath} + y_M\vec{\jmath} \Rightarrow \overrightarrow{OM}/cart = \rho\cos\theta\vec{\imath} + \rho\sin\theta\vec{\jmath}$

$\overrightarrow{OM}/pol = \rho\vec{U}_r$ and $\overrightarrow{OM}/cart = \rho(\cos\theta\vec{\imath} + \sin\theta\vec{\jmath})$

By identification: $\vec{u_r} = \cos\theta\vec{\imath} + \sin\theta\vec{\jmath}$

Rule: Note: The derivative of a unit vector with respect to an angle is a unit vector perpendicular to the angle in the positive direction.

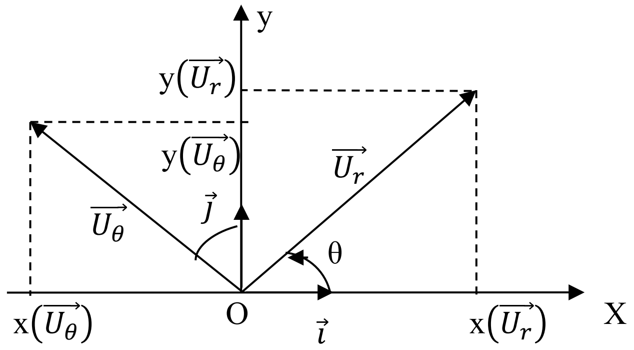
مشتقة شعاع وحدة بالنسبة إلى الزاوية هي شعاع وحدة عمودي على هذا الاخير في الاتجاه الموجب

The vector $\vec{u_\theta} \perp \vec{u_r}$ in the direction of θ which corresponds to the direct direction therefore

$$\vec{u_\theta} = \frac{d\vec{u_r}}{d\theta}$$

So $\vec{u_\theta} = -\sin\theta\vec{\imath} + \cos\theta\vec{\jmath}$

By projecting the unit vectors we will have the same results



$\overrightarrow{u_r} = x(\overrightarrow{u_r})\vec{i} + y(\overrightarrow{u_r})\vec{j} \Rightarrow \overrightarrow{u_r} = |\overrightarrow{u_r}|\cos\theta\vec{i} + |\overrightarrow{u_r}|\sin\theta\vec{j}$

with $|\overrightarrow{u_r}| = 1$ since it is a unit vector therefore $\overrightarrow{u_r} = 1\cos\theta\vec{i} + 1\sin\theta\vec{j}$

$\overrightarrow{u_\theta} = x(\overrightarrow{u_\theta})\vec{i} + y(\overrightarrow{u_\theta})\vec{j} \Rightarrow \overrightarrow{u_\theta} = -|\overrightarrow{u_\theta}|\sin\theta\vec{i} + |\overrightarrow{u_\theta}|\cos\theta\vec{j}$

with $|\overrightarrow{u_\theta}| = 1$ since it is a unit vector therefore $\overrightarrow{u_\theta} = -\sin\theta\vec{i} + \cos\theta\vec{j}$

To write the unit vectors $\overrightarrow{u_r}$ and $\overrightarrow{u_\theta}$ as a function of $\vec{i}$ and $\vec{j}$ we use the passage table

	$\vec{i}$	$\vec{j}$
$\overrightarrow{u_r}$	cosθ	$\sin\theta$
$\overrightarrow{u_\theta}$	- $\sin\theta$	cosθ

$\vec{i} = \cos\theta\overrightarrow{u_r} - \sin\theta\overrightarrow{u_\theta}$ and $\vec{j} = \sin\theta\overrightarrow{u_r} + cos\theta\overrightarrow{u_\theta}$

Example :

Write the vector $\vec{A} = 2x\vec{i} + y\vec{j}$

$\begin{cases} x = \rho\, cos\theta \\ y = \rho\, sin\theta \end{cases}$ and $\begin{cases} \vec{i} = \cos\theta\overrightarrow{u_r} - \sin\theta\overrightarrow{u_\theta} \\ \vec{j} = \sin\theta\overrightarrow{u_r} + cos\theta\overrightarrow{u_\theta} \end{cases}$

So $\vec{A} = 2\,\rho\, cos\theta\,(\cos\theta\overrightarrow{u_r} - \sin\theta\overrightarrow{u_\theta}) + \rho\, sin\theta\,(\sin\theta\overrightarrow{u_r} + \cos\theta\overrightarrow{u_\theta})$

$\Rightarrow \vec{A} = 2\,\rho\, cos^2\theta\overrightarrow{u_r} - 2\rho(\cos\theta\sin\theta)\overrightarrow{u_\theta} + \rho\, sin^2\theta\,\overrightarrow{u_r} + \rho(\sin\theta\cos\theta)\overrightarrow{u_\theta}$

$\Rightarrow \vec{A} = \rho\,(2cos^2\theta + sin^2\theta)\overrightarrow{u_r} - \rho(\cos\theta\sin\theta)\overrightarrow{u_\theta}$

$\Rightarrow \vec{A} = (\rho\, cos^2\theta + 1)\overrightarrow{u_r} - \rho(\cos\theta\sin\theta)\overrightarrow{u_\theta}$

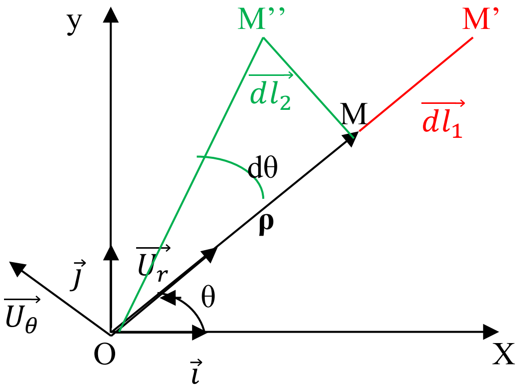
- **Elementary displacement**

The variables ρ and θ are independent: we fix one and change the other

- We fix θ, and we change ρ, then the moving point moves from the point M(ρ, θ) to the point M'(ρ+dρ, θ)

$$\overrightarrow{dl_1} = \overrightarrow{MM'} = d\rho\overrightarrow{u_r}$$

- We fix ρ, and we change θ, then the moving point moves from the point M(ρ, θ) to the point M'(ρ, θ+d θ)



The angle θ varies by dθ, causing a linear displacement of point M towards point M" $\overrightarrow{U_\theta}$, $\left(\overrightarrow{MM''} \perp \overrightarrow{u_\theta}\right)$

In the right triangle OMM'', MM'' = ρsinθdθ.

Since dθ is very small, we can approximate Sin(dθ) as dθ.

Therefore, MM'' = ρdθ, so

$$\overrightarrow{dl_2} = \overrightarrow{MM''} = \rho d\theta\overrightarrow{u_\theta}$$

so

$$\overrightarrow{dl} = \overrightarrow{dl_1} + \overrightarrow{dl_2} = d\rho\overrightarrow{u_r} + \rho d\theta\overrightarrow{u_\theta}$$

We can obtain the same result mathematically:

$$\overrightarrow{OM} = \rho\vec{U}_r \Rightarrow d\overrightarrow{OM} = d\rho\vec{U}_r + \rho d\vec{U}_r$$

To make the derivative of a unit vector $d\vec{U}_r$, we must bring out the derivative with respect to an angle $\frac{d\vec{U}_r}{d\theta}$ for this we multiply and divide by dθ

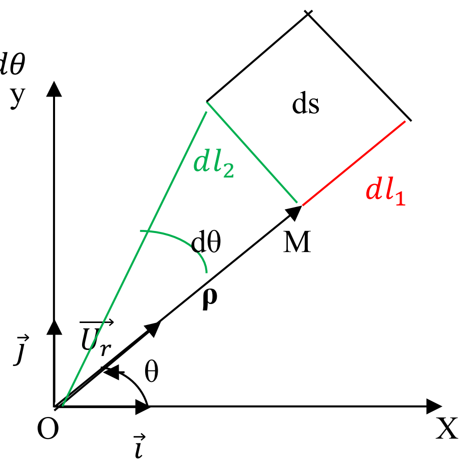
$$d\vec{U}_r = \frac{d\vec{U}_r}{d\theta}d\theta = \vec{U}_\theta d\theta$$

With $\frac{d\vec{U}_r}{d\theta} = \vec{U}_\theta$ so $d\overrightarrow{OM} = d\rho\vec{U}_r + \rho\, d\theta\vec{U}_\theta$

Calculation of the surface:

$$ds = \left|\overrightarrow{dl_1}\right|.\left|\overrightarrow{dl_2}\right| = d\rho.\rho d\theta \Rightarrow s = \iint d\rho.\rho d\theta$$

We can separate the variables since they are independent



$s = \int_0^R \rho d\rho \,.\int_0^{2\pi} d\theta = \frac{R^2}{2} 2\pi$

with ρ varies from 0 to R and θ varies from 0 to $2\pi \Rightarrow s = \pi R^2$

- **Velocity vector**

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{d\rho}{dt}\vec{U}_r + \rho\,\frac{d\vec{U}_r}{dt}$$

We have: $\frac{d\vec{U}_r}{dt} = \frac{d\vec{U}_r}{dt}\frac{d\theta}{d\theta} = \frac{d\vec{U}_r}{d\theta}\frac{d\theta}{dt}$

With : $\frac{d\vec{U}_r}{d\theta} = \vec{U}_\theta$ donc $\frac{d\vec{U}_r}{dt} = \frac{d\theta}{dt}\vec{U}_\theta$ so $\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{d\rho}{dt}\vec{U}_r + \rho\frac{d\theta}{dt}\vec{U}_\theta$

$\Rightarrow \vec{v} = \rho^{\cdot}\vec{U}_r + \rho\theta^{\cdot}\vec{U}_\theta$ with $\rho^{\cdot} = \frac{d\rho}{dt}$ and $\theta^{\cdot} = \frac{d\theta}{dt}$

- **Acceleration vector**

$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\overrightarrow{OM}}{dt^2} = \frac{d^2\rho}{dt^2}\vec{U}_r + \frac{d\rho}{dt}\frac{d\vec{U}_r}{dt} + \frac{d\rho}{dt}\frac{d\theta}{dt}\vec{U}_\theta + \rho\,\frac{d^2\theta}{dt^2}\vec{U}_\theta + \rho\,\frac{d\theta}{dt}\frac{d\vec{U}_\theta}{dt}$

$$\Rightarrow \vec{a} = \frac{d^2\rho}{dt^2}\vec{U}_r + \frac{d\rho}{dt}\frac{d\theta}{dt}\vec{U}_\theta + \frac{d\rho}{dt}\frac{d\theta}{dt}\vec{U}_\theta + \rho\,\frac{d^2\theta}{dt^2}\vec{U}_\theta - \rho\left(\frac{d\theta}{dt}\right)^2\vec{U}_r$$

With : $\frac{d\vec{U}_r}{d\theta} = \vec{U}_\theta et \frac{d\vec{U}_\theta}{d\theta} = -\vec{U}_r$

$$\text{So : } \vec{a} = \rho^{\cdot\cdot}\vec{U}_r + 2\rho^{\cdot}\theta^{\cdot}\vec{U}_\theta + \rho\theta^{\cdot\cdot}\vec{U}_\theta - \rho(\theta^{\cdot})^2\vec{U}_r$$

4.3. Cylindrical Coordinates الاحداثيات الاسطوانية

If the spatial trajectory involves ρ and z playing a specific role in determining the position vector ($\overrightarrow{OM}$); for example, the movement of air molecules in a whirlwind; it is preferable to use cylindrical coordinates (ρ, θ, z). With:

ρ: polar radius

θ: polar angle

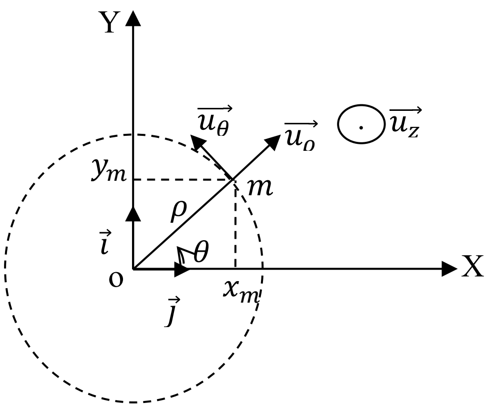
z: altitude or height

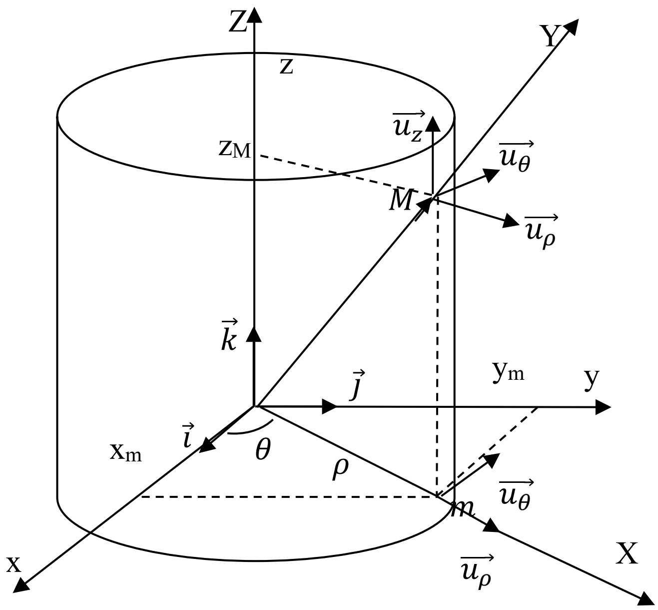
and $\begin{cases} \rho = |\overrightarrow{Om}|, \ 0 < \rho < R \\ \theta = \left((ox), \overrightarrow{Om}\right), 0 < \theta < 2\pi \\ z = z_M\,,\ 0 < z < H \end{cases}$

Where m is the projection of point M onto the plane (Oxy), and R is the radius of the cylinder, and H is the height of the cylinder.

If we add the 'z' component to polar coordinates in space, we obtain what is known as cylindrical coordinates. Consider R(Oxyz) and a point M belonging to a cylinder.

Point M is identified by three coordinates ρ, θ (polar coordinates), and z.





- **Position vector**

The position vector in cylindrical coordinates (ρ, θ, z) in the orthonormal frame R'$(O,\overrightarrow{u_\rho},\overrightarrow{u_\theta},\overrightarrow{u_z})$ is written:

$$\begin{cases} \vec{r} = \overrightarrow{OM} = \overrightarrow{Om} + \overrightarrow{mM}\ (Relation\ de\ \text{Charles}) \\ \overrightarrow{Om} = \rho\overrightarrow{u_\rho}\ (Coordonnées\ polaires) \\ \overrightarrow{mM} = z\overrightarrow{u_z}\ (\ hauteur\ \ du\ cylindre) \end{cases} \Rightarrow \vec{r} = \overrightarrow{OM} = \rho\overrightarrow{u_\rho} + z\overrightarrow{u_z}$$

- **Unit vectors**

The unit vectors $\vec{U}_\rho$ is following $\overrightarrow{Om}$ (m is the projection of the point M on the plane (Oxy)) and $\vec{U}_\theta$ is perpendicular to $\vec{U}_r$ and $\overrightarrow{Om}$ in the direction of θ **($\vec{U}_\rho \perp \vec{U}_\theta$)** and $\vec{U}_z$ is following (Oz), **($\vec{U}_z \parallel \vec{k}$)** and it is perpendicular to the plane formed by the two other unit vectors ($\vec{U}_\rho$ and $\vec{U}_\theta$).

Transit relations between cylindrical coordinates and Cartesian coordinates:

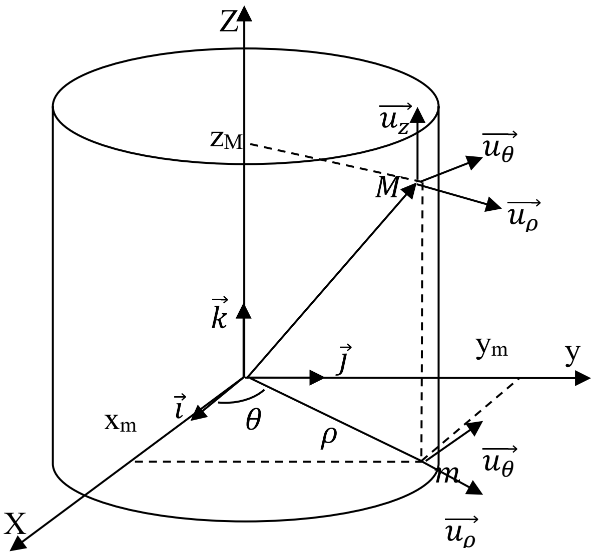
By projecting the point m onto the axes (Ox) and (Oy) (like polar coordinates) z is the height

$$\begin{cases} \boldsymbol{x} = \rho\ cos\theta \\ \boldsymbol{y} = \rho\ sin\ \theta \\ \boldsymbol{z} = \boldsymbol{z} \end{cases}$$

$$\overrightarrow{OM}\Big/_{cylin} = \overrightarrow{Om} + \overrightarrow{mM} = \rho\overrightarrow{u_\rho} + z\overrightarrow{u_z}$$

$\overrightarrow{OM}/_{cart} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$

$\overrightarrow{OM}/_{cart} = \rho\,(cos\theta\,\vec{\imath} + sin\,\theta\,\vec{\jmath}) + z\vec{k}$



By identification

$$\begin{cases} \overrightarrow{u_\rho} = cos\theta\,\vec{\imath} + sin\,\theta\,\vec{\jmath} \\ \overrightarrow{u_\theta} = \dfrac{d\overrightarrow{u_\rho}}{d\theta} = -sin\theta\,\vec{\imath} + cos\,\theta\,\vec{\jmath} \\ \overrightarrow{u_z} = \vec{k} \end{cases}$$

Using the passage table :

	$\vec{\imath}$	$\vec{\jmath}$	$\vec{k}$
$\overrightarrow{u_\rho}$	Cosθ	Sinθ	0
$\overrightarrow{u_\theta}$	- sinθ	Cosθ	0
$\overrightarrow{u_z}$	0	0	1

$\vec{\imath} = \cos\theta\overrightarrow{u_\rho} - \sin\theta\overrightarrow{u_\theta}$

$\vec{\jmath} = \sin\theta\overrightarrow{u_\rho} + cos\theta\overrightarrow{u_\theta}$ and $\vec{k} = \overrightarrow{u_z}$

- **Elementary displacement**

The variables ρ, θ and z are independent: we fix one and change the other

- We fix θ, z and we change ρ, then the moving point moves from the point M(ρ, θ,z) to the point M'(ρ+dρ, θ,z)

$$\overrightarrow{dl_1} = \overrightarrow{MM'} = d\rho\overrightarrow{u_\rho}$$

- We fix ρ, z and we change θ, then the moving point moves from the point M(ρ, θ,z) to the point M'(ρ, θ+d θ,z)

The angle θ varies by dθ, this leads to a linear movement from point M towards point M'' following $\overrightarrow{U_\theta}$, $\left(\overrightarrow{MM''} \parallel \overrightarrow{u_\theta}\right)$

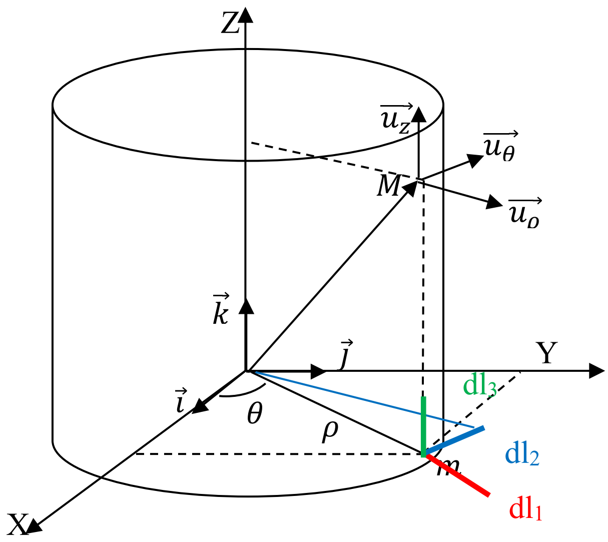
In the right triangle OMM'', MM''=ρ sindθ

dθ is so small then sin dθ≈dθ.

Then MM''=ρdθ therefore $\overrightarrow{dl_2} = \overrightarrow{MM''} = \rho d\theta\overrightarrow{u_\theta}$

- We fix ρ, θ and we change z, then the moving point moves from the point M(ρ, θ,z) to the point M'''(ρ, θ,z+dz)

$$\overrightarrow{dl_3} = \overrightarrow{MM'''} = dz\overrightarrow{u_z}$$



So

$$\overrightarrow{dl} = \overrightarrow{dl_1} + \overrightarrow{dl_2} + \overrightarrow{dl_3} = d\rho\overrightarrow{u_\rho} + \rho d\theta\overrightarrow{u_\theta} + dz\overrightarrow{u_z}$$

We can obtain the same result mathematically

$\overrightarrow{OM} = \rho\overrightarrow{u_\rho} + z\overrightarrow{u_z} \Rightarrow d\overrightarrow{OM} = d\rho\vec{U}_\rho + \rho d\vec{U}_\rho + dz\overrightarrow{u_z} + zd\overrightarrow{u_z}$

$d\overrightarrow{u_z} = 0$ car $\overrightarrow{u_z} = \vec{k}$ it's a vector fix.

$$d\vec{U}_\rho = \frac{d\vec{U}_\rho}{d\theta}\, d\theta = \vec{U}_\theta d\theta$$

With $\frac{d\vec{U}_\rho}{d\theta} = \vec{U}_\theta$ so $d\overrightarrow{OM} = d\rho\vec{U}_\rho + \rho\, d\theta\vec{U}_\theta + dz\overrightarrow{u_z}$

- **The cylinder volume**

$$dV = |\overrightarrow{dl_1}|.|\overrightarrow{dl_2}|.|\overrightarrow{dl_3}| = d\rho.\rho d\theta.dz \Rightarrow V = \iiint d\rho.\rho d\theta \text{ dz}$$

We can separate the variables since they are independent

$V = \int_0^R \rho d\rho \,.\int_0^{2\pi} d\theta \int_0^H dz = \frac{R^2}{2} 2\pi H \Rightarrow V = \pi R^2 H$

(with ρ varies from 0 to R and θ varies from 0 to 2π and z varies from 0 to H)

- **The surface of the base of cylinder**

$$ds_{base} = |\overrightarrow{dl_1}|.|\overrightarrow{dl_2}| = d\rho.\rho d\theta \Rightarrow s_{base} = \iint d\rho.\rho d\theta$$

We can separate the variables since they are independent

$$s = \int_0^R \rho d\rho\,.\int_0^{2\pi} d\theta = \frac{R^2}{2}2\pi \Rightarrow s_{base} = \pi R^2$$

- **The lateral surface of cylinder**

$$ds_{lat} = |\overrightarrow{dl_2}|.|\overrightarrow{dl_3}| = d\rho.\rho d\theta \Rightarrow s_{base} = \iint \rho d\theta.dz$$

The radius is constant ρ=R, the variables are independent so we can separate them

$$s = R\int_0^{2\pi} d\theta\,.\int_0^{H} dz = R2\pi H \Rightarrow s_{base} = 2\pi RH$$

- **Velocity vector**

The velocity in this case is written by: $\vec{\boldsymbol{v}} = \frac{d\overrightarrow{OM}}{dt} = \frac{d\rho}{dt}\vec{U}_r + \rho\frac{d\vec{U}_r}{dt} + \frac{dz}{dt}\vec{U}_z + z\frac{d\vec{U}_z}{dt}$

$$\frac{d\vec{U}_r}{dt} = \frac{d\vec{U}_r}{dt}\frac{d\theta}{d\theta} = \frac{d\vec{U}_r}{d\theta}\frac{d\theta}{dt}$$

With: $\frac{d\vec{U}_r}{d\theta} = \vec{U}_\theta$ so $\frac{d\vec{U}_r}{dt} = \frac{d\theta}{dt}\vec{U}_\theta$ and $\frac{d\vec{U}_z}{dt} = \vec{O}$

$$\vec{\boldsymbol{v}} = \frac{d\overrightarrow{OM}}{dt} = \frac{d\rho}{dt}\vec{U}_r + \rho\,\frac{d\theta}{dt}\vec{U}_\theta + \frac{dz}{dt}\vec{U}_z$$

$$\Rightarrow \vec{\boldsymbol{v}} = \rho^{\cdot}\vec{U}_r + \rho\,\theta^{\cdot}\vec{U}_\theta + z^{\cdot}\overrightarrow{U_z}$$

With: $\rho^{\cdot} = \frac{d\rho}{dt}$, $\theta^{\cdot} = \frac{d\theta}{dt}$ and $z^{\cdot} = \frac{dz}{dt}$

- **Acceleration vector**

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\overrightarrow{OM}}{dt^2} = \frac{d^2\rho}{dt^2}\vec{U}_r + \frac{d\rho}{dt}\frac{d\vec{U}_r}{dt} + \frac{d\rho}{dt}\frac{d\theta}{dt}\vec{U}_\theta + \rho\frac{d^2\theta}{dt^2}\vec{U}_\theta + \rho\frac{d\theta}{dt}\frac{d\vec{U}_\theta}{dt} + \frac{d^2z}{dt^2}\vec{U}_z + \frac{dz}{dt}\frac{d\vec{U}_z}{dt}$$

$$\Rightarrow \vec{a} = \frac{d^2\rho}{dt^2}\vec{U}_r + \frac{d\rho}{dt}\frac{d\theta}{dt}\vec{U}_\theta + \frac{d\rho}{dt}\frac{d\theta}{dt}\vec{U}_\theta + \rho\,\frac{d^2\theta}{dt^2}\vec{U}_\theta - \rho\left(\frac{d\theta}{dt}\right)^2\vec{U}_r + \frac{d^2z}{dt^2}\vec{U}_z$$

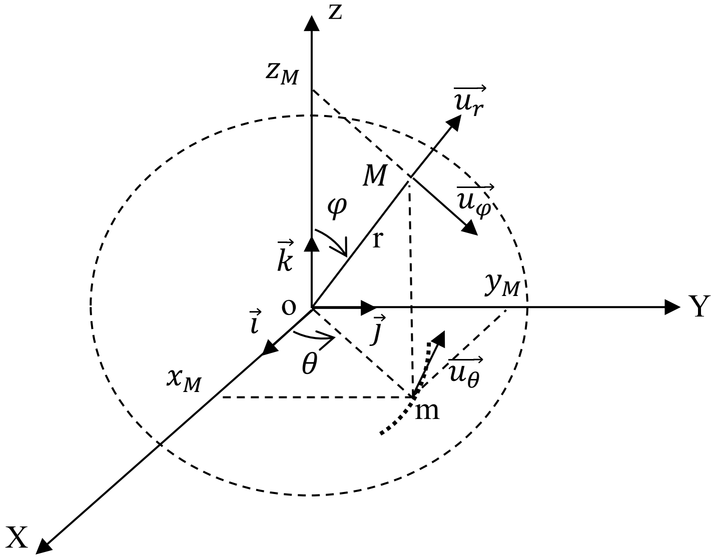
With : $\frac{d\vec{U}_r}{d\theta} = \vec{U}_\theta\,, \frac{d\vec{U}_\theta}{d\theta} = -\vec{U}_r\ and\ \frac{d\vec{U}_z}{dt} = \vec{O}$

$$\Rightarrow\ \vec{a} = \rho^{\cdot\cdot}\vec{U}_r + 2\rho^{\cdot}\theta^{\cdot}\vec{U}_\theta + \rho\theta^{\cdot\cdot}\vec{U}_\theta - \rho(\theta^{\cdot})^2\vec{U}_r + z^{\cdot\cdot}\overrightarrow{U_z}$$

4.4. Spherical coordinates الاحداثيات الكروية

When the point O and the distance r separating M and O play a characteristic role, the use of spherical coordinates (r,θ,φ) are the best suited in the orthonormed base ($\overrightarrow{u_r}, \overrightarrow{u_\theta}, \overrightarrow{u_\varphi}$) with:

$$\begin{cases} r = \left|\overrightarrow{OM}\right|, \ 0 < r < R \\ \theta = \left((ox), \overrightarrow{Om}\right) \ 0 < \theta < 2\pi \\ \varphi = \left((oz), \overrightarrow{OM}\right) \ 0 < \varphi < \pi \end{cases}$$



With m is the projection of M in the plane (Oxy).

- **Position Vector** :

The position vector in spherical coordinates (r,θ,φ) is written as: $\boldsymbol{\vec{r}} = \boldsymbol{\overrightarrow{OM}} = \boldsymbol{r\overrightarrow{U_r}}$

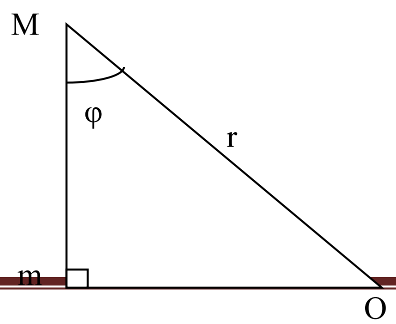
- **The unit vectors**

The unit vectors $\boldsymbol{\overrightarrow{U_r}}$ is following $\overrightarrow{OM}$ and $\boldsymbol{\overrightarrow{U_\varphi}}$ is perpendicular to $\overrightarrow{U_r}$ and $\overrightarrow{OM}$ in the direction of φ **($\boldsymbol{\overrightarrow{U_\varphi}} \perp \boldsymbol{\overrightarrow{U_r}}$)** and $\boldsymbol{\overrightarrow{U_\theta}}$ is perpendicular to $\overrightarrow{Om}$ **($\boldsymbol{\overrightarrow{U_\theta}} \perp \overrightarrow{Om}$).**

Transit relations between spherical coordinates and Cartesian coordinates

By projecting m onto the axes (Ox) and (Oy)

$$\begin{cases} \boldsymbol{x} = \left|\overrightarrow{Om}\right| cos\theta \\ \boldsymbol{y} = \left|\overrightarrow{Om}\right| \ sin\,\theta \\ \boldsymbol{z} = \left|\overrightarrow{mM}\right| \end{cases}$$



By taking the right triangle (OmM):

We have Om=r sinφ and mM=r cosφ, replacing them

in passing relationships we will have:

$$\begin{cases} \boldsymbol{x} = r\,sin\varphi\;cos\theta \\ \boldsymbol{y} = r\,sin\varphi\;\;sin\,\theta \\ \boldsymbol{z} = r\cos\varphi \end{cases}$$

$$\overrightarrow{OM}\Big/_{sph} = r\overrightarrow{u_r}$$

$$\overrightarrow{OM}/_{cart} = r\,sin\varphi\;cos\theta\vec{\imath} + r\,sin\varphi\;\;sin\,\theta\vec{\jmath} + r\cos\varphi\,\vec{k}$$

$$\overrightarrow{OM}/_{cart} = r\left(\,sin\varphi\;cos\theta\vec{\imath} + \;sin\varphi\;\;sin\,\theta\vec{\jmath} + \cos\varphi\,\vec{k}\right)$$

By identification

$$\overrightarrow{u_r} = sin\varphi\;cos\theta\vec{\imath} + \;sin\varphi\;\;sin\,\theta\vec{\jmath} + \cos\varphi\,\vec{k}$$

$$\overrightarrow{u_\varphi} = \frac{-d\overrightarrow{U_r}}{d(-\varphi)} = \frac{d\overrightarrow{U_r}}{d\varphi} = \;cos\varphi\;cos\theta\vec{\imath} + cos\;\varphi\;\;sin\,\theta\vec{\jmath} - \sin\varphi\,\vec{k}$$

$$\overrightarrow{U_\theta} = \overrightarrow{U_r}\Lambda\overrightarrow{U_\varphi} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ sin\varphi\;cos\theta & sin\varphi\;\;sin\,\theta & \cos\varphi \\ cos\varphi\;cos\theta & cos\;\varphi\;\;sin\,\theta & -\sin\varphi \end{vmatrix}$$

$$= \vec{\imath}(-sin^2\varphi\;\;sin\,\theta - cos\,^2\varphi\;\;sin\,\theta) - \vec{\jmath}(-sin^2\varphi\;cos\theta - cos^2\varphi\;cos\theta)$$

$$+ \vec{k}(sin\varphi\;cos\theta cos\;\varphi\;\;sin\,\theta - sin\varphi\;\;sin\,\theta cos\varphi\;cos\theta)$$

$$\overrightarrow{U_\theta} = -\sin\theta\vec{\imath} + cos\theta\vec{\jmath}$$

By using the pasage table :

	$\vec{\imath}$	$\vec{\jmath}$	$\vec{k}$
$\overrightarrow{u_r}$	$sin\varphi\;cos\theta$	$sin\varphi\;\;sin\,\theta$	$\cos\varphi$
$\overrightarrow{u_\varphi}$	$cos\varphi\;cos\theta$	$cos\;\varphi\;\;sin\,\theta$	-sin φ
$\overrightarrow{u_\theta}$	$-\sin\theta$	$cos\theta$	1

$$\vec{\imath} = \;sin\varphi\;cos\theta\,\overrightarrow{u_r} + cos\varphi\;cos\theta\,\overrightarrow{u_\varphi} - \sin\theta\,\overrightarrow{u_\theta}$$
$$\vec{\jmath} = \;sin\varphi\;sin\theta\,\overrightarrow{u_r} + cos\varphi\;sin\theta\,\overrightarrow{u_\varphi} + \cos\theta\,\overrightarrow{u_\theta}$$
$$\vec{k} = \;cos\varphi\;\;\overrightarrow{u_r} - \;sin\varphi\,\overrightarrow{u_\varphi}$$

- **Elementary displacement**

The variables r, φ and θ are independent: we fix one and change the other:

- We fix φ,θ and we change r, then the moving point moves from the point M(r, φ,θ) to the point M'(r+dr, φ,θ) so $\overrightarrow{dl_1} = \overrightarrow{MM'} = dr\overrightarrow{u_r}$
- We fix r, θ and we change φ, then the moving point moves from the point M(r, φ ,θ) to the point M''(r, , φ+dφ ,θ)

The angle φ varies by dφ, this leads to a linear movement from point M towards point M" following $\overrightarrow{U_\varphi}$, $\left(\overrightarrow{MM''} \perp \overrightarrow{u_\varphi}\right)$.

In the right triangle OMM'', MM''= r.sindφ

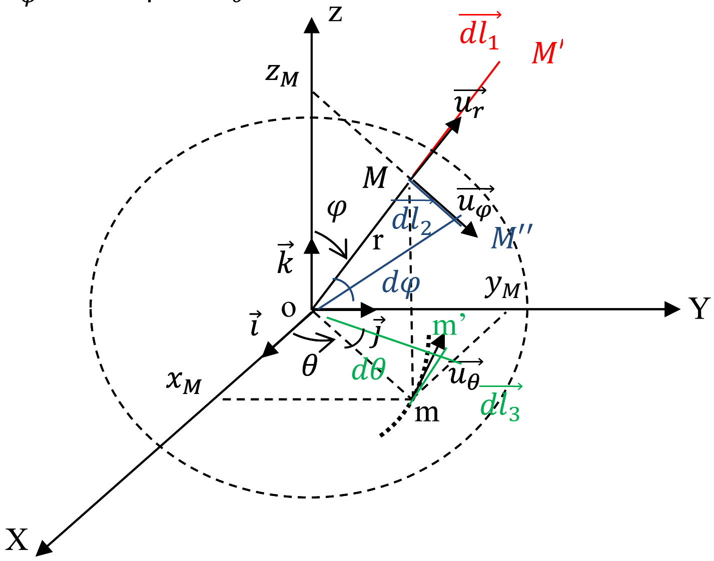
dφ is so small then sin dφ≈dφ

then MM"=r.dφ therefore $\overrightarrow{dl_2} = \overrightarrow{MM''} = r\, d\varphi\, \overrightarrow{u_\varphi}$

- We fix r, φ and we change θ, then the moving point moves from the point M(r, φ, θ) to the point M''(r, φ, θ+dθ)

The angle φ varies by dφ, this leads to a linear displacement of the point m (projection of the point M in the plane (Oxy)) towards the following point m' $\overrightarrow{U_\theta}$, $\left(\overrightarrow{mm'} \perp \overrightarrow{u_\theta}\right)$.
In the right triangle Omm', mm'= Om.sindθ
dθ is so small then sin dθ≈dθ
So *mm'*=Omdθ and Om=r sinφ therefore $\overrightarrow{dl_2} = \overrightarrow{mm'} = \mathrm{r}\sin\varphi\, d\theta\, \overrightarrow{u_\theta}$

$\overrightarrow{dOM} = dr\overrightarrow{u_r} + rd\varphi\overrightarrow{u_\varphi} + \mathrm{r}\sin\varphi\, d\theta\, \overrightarrow{u_\theta}$



Or mathematically :

$$\overrightarrow{OM} = r\overrightarrow{U_r} \Rightarrow d\overrightarrow{OM} = d\left(r\overrightarrow{U_r}\right) = dr\overrightarrow{U_r} + rd\overrightarrow{U_r}$$

$$d\overrightarrow{U_r} = \frac{\partial U_r}{\partial \theta} d\theta + \frac{\partial U_r}{\partial \varphi}\, d\varphi$$

$$\overrightarrow{U_r} = sin\varphi\ cos\theta\vec{\imath} + \ sin\varphi\ \ sin\ \theta\vec{\jmath} + \cos\varphi\,\vec{k}$$

$$\frac{\partial\overrightarrow{U_r}}{\partial\theta} = -sin\varphi\ sin\theta\vec{i} + \ sin\varphi\cos\theta\vec{j} = \sin\varphi\left(-sin\theta\vec{i} + \ \ cos\ \theta\vec{j}\right)$$

$$\Rightarrow \frac{\partial\overrightarrow{U_r}}{\partial\theta} = sin\varphi\overrightarrow{U_\theta}$$

$$\frac{\partial\overrightarrow{U_r}}{\partial\varphi} = cos\varphi\ cos\theta\vec{i} + cos\varphi\ \sin\theta\vec{j} - sin\varphi\vec{k}$$

$$\Rightarrow \frac{\partial\overrightarrow{U_r}}{\partial\varphi} = \overrightarrow{U_\varphi}$$

$$d\overrightarrow{OM} = dr\overrightarrow{U_r} + rd\varphi\overrightarrow{U_\varphi} + rsin\varphi d\theta\overrightarrow{U_\theta}$$

- **Volume of sphere**

$dV=dl_1dl_2dl_3=dr\ \ r\ sin\varphi\ d\theta\ \ r\ d\varphi \Rightarrow V\ = \iiint r^2 dr\sin\varphi\ d\varphi\ d\theta$

$$\Rightarrow V = \int_0^R r^2 dr \int_0^\pi \sin\varphi \int_0^{2\pi} d\theta = \left.\frac{r^3}{3}\right]\left(-\ cos\varphi)\right]\ \ \theta]$$

$$\Rightarrow V = \frac{4}{3}\ \pi R^3$$

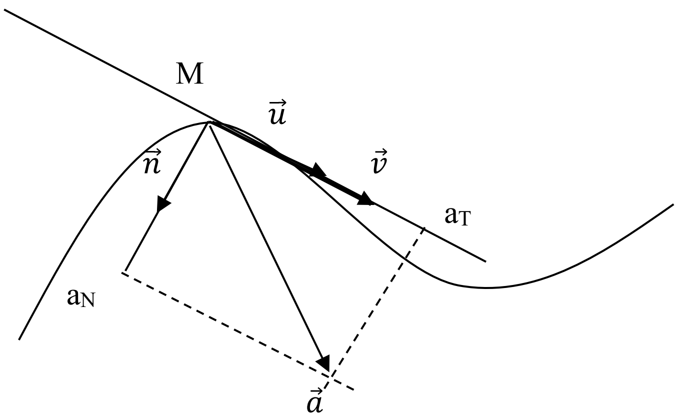
- **Velocity vector**

The velocity vector is written in spherical coordinates (r,θ,φ) by:

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{dr}{dt}\overrightarrow{U_r} + r\frac{d\varphi}{dt}\overrightarrow{U_\varphi} + rsin\varphi\frac{d\theta}{dt}\overrightarrow{U_\theta}$$

4.5. Intrinsic coordinates (Frenet frame) احداثيات الحركة المنحنية

We used to work in a fixed frame, but in this case, we study the motion in a moving frame that travels with the moving point "M". This frame is the Frenet frame.



We study the motion in the Frenet frame:

The Frenet frame is a two-dimensional reference frame.

- $\vec{u}$ is the unit vector along the tangent to the trajectory.
- $\vec{n}$ is the unit vector normal to the trajectory and perpendicular to $\vec{u}$, directed towards the center of curvature.
- The position remains unchanged (the frame moves with point M).
- The velocity vector is tangent to the trajectory, and it is written as: $\vec{\boldsymbol{v}} = |\vec{\boldsymbol{v}}|\vec{\boldsymbol{u}}$
- The acceleration vector :

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d|\vec{v}|\vec{u}}{dt} = \frac{d|\vec{v}|}{dt}\vec{u} + |\vec{v}|\frac{d\vec{u}}{dt}$$

$$\frac{d\vec{u}}{dt} = \frac{d\vec{u}}{d\theta}.\frac{d\theta}{dt} = \vec{n}.\omega with \quad \vec{n} = \frac{d\vec{u}}{d\theta} and\ \omega = \frac{d\theta}{dt}$$

The acceleration vector is written by: $\vec{a} = a_T\vec{u} + a_N\vec{n}$

So : $\vec{a} = \frac{d|\vec{v}|}{dt}\vec{u} + |\vec{v}|.\vec{n}.\omega$

(the perimeter of a circle (محيط دائرة) $l = 2\pi R$, for the length of a segment (طول قوس)

$x = \theta R$; from angular velocity to linear velocity by $\frac{dx}{dt} = R\frac{d\theta}{dt} \Rightarrow v = R\omega$)

Hence:

$\omega = \frac{v}{R}$ with R is the radius of the curvature of the trajectory.

So $\vec{a} = \frac{d|\vec{v}|}{dt}\vec{u} + \frac{v^2}{R}\vec{n}$

The normal acceleration (التسارع الناظمي) and tangential acceleration (التسارع المماس) are written by :
$$\begin{cases} a_T = \frac{d|\vec{v}|}{dt} \\ a_N = \frac{v^2}{R} \end{cases}$$

$$|\vec{a}| = \sqrt{a_x^2 + a_y^2} = \sqrt{a_N^2 + a_T^2}$$

<u>Note :</u>

- $R \to \infty$: so the trajectory is a line .
- R is constant: so the trajectory is circular.

5. Study of some movements

5.1. Rectilinear motion حركة خطية

We have linear motion if the trajectory is a straight line.

We choose a point O as the origin on the trajectory and a unit vector $\vec{\imath}$.

The position of the mobile M, as a function of time, is identified by its abscissa:

$x(t) = \overline{OM(t)}$.

The position vector will be: $\overrightarrow{r(t)} = \overrightarrow{OM(t)} = x(t)\vec{\imath}$

5.1.1. Uniform rectilinear motion حركة مستقيمة منتظمة URM

We have uniform rectilinear motion if the trajectory is a straight line and the velocity vector is constant. This is a motion with zero acceleration $\overrightarrow{a(t)} = \vec{0}$.

The initial conditions to t=0 ; x=x_0.

- ***The velocity***

$$a = \frac{dv}{dt} = 0 \Rightarrow \int_{v_0}^{v} dv = \int_0^t 0.dt = cte$$

So ***$v=v_0=cte$***

- ***The position***

$$v = \frac{dx}{dt} = v_0 \Rightarrow \int_{x_0}^{x} dx = \int_0^t v_0 d\,t = [v_0 t]_0^t = v_0 t$$

So : ***$x=v_0 t+x_0$ This is the hourly equation of the motion. URM***

5.1.2. Uniformly varied rectilinear motion حركة مستقيمة متغيرة بانتظام UVRM

One has a uniformly varied rectilinear movement if the trajectory is a straight and the acceleration is constant.

The initial conditions to t=0 ; $v=v_0$ and x=x_0

- ***The velocity***

$$a = \frac{dv}{dt} = a_0 \Rightarrow \int_{v_0}^{v} dv = \int_0^t a_0 d\,t = [a_0 t]_0^t$$

So ***$v=a_0 t+v_0$***

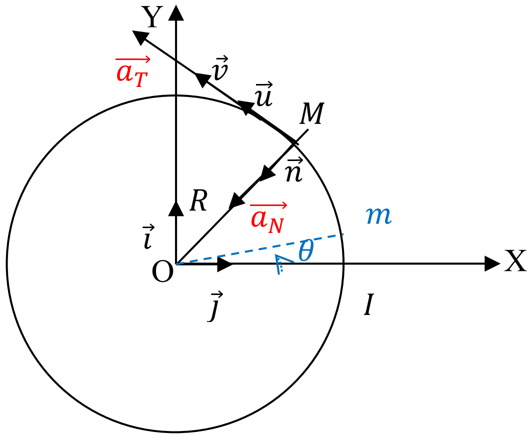
- ***The position***

$$v = \frac{dx}{dt} = a_0 t + v_0 \quad \Rightarrow \int_{x_0}^{x} dx = \int_0^t (a_0 t + v_0) d\,t = \left[\frac{1}{2} a_0 t^2 + v_0 t\right]_0^t$$

So $\boldsymbol{x = \frac{1}{2} a_0 t^2 + v_0 t + x_0}$ ***this is the hourly equation of the motion UVRM***

5.2. Circular motion حركة دائرية

Circular motion is plane motion with constant radius of curvature ρ=R. The trajectory of the moving object is a circle of radius R .



- ***The position***

The moving point travels from point I to point M, thus the trajectory forms an arc$\widehat{IM}$.

By considering an elementary displacement of the moving point from point I to point m, we would have a displacement in the form of an elementary arc Im.

In the right triangle OIm, $\widehat{Im}$=R sinθ

In the right triangle. If θ is so small then$sin\theta \approx \theta$.

So $\boldsymbol{\widehat{Im}=R.\theta}$

- ***The speed***

$$v = \frac{d\widehat{Im}}{dt} = R\frac{d\theta}{dt}$$

R is constant, the speed is following the trajectory, so it is written $\vec{v} = v\vec{u}$ so the vector $\vec{u}$ would be following the tangent.

$\boldsymbol{\frac{d\theta}{dt} = \dot{\theta} = \omega}$ ***is the angular velocity*** **السرعة الزاوية**

$$v = R\frac{d\theta}{dt} = R.\boldsymbol{\dot{\theta}} = \boldsymbol{R.\omega}$$

Note: The relationship between linear velocity and angular velocity is: $\mathbf{v = R\omega}$

- ***The acceleration***

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv}{dt}\vec{u} + v\frac{d\vec{u}}{dt}$$

$\frac{d\vec{u}}{dt} = \frac{d\vec{u}}{d\theta}\frac{d\theta}{dt}$ $with$ $\frac{d\vec{u}}{d\theta} = \vec{n}$

(with$(\vec{u}, \vec{n})$ the unit vectors in the Fresnet farme and $\frac{d\theta}{dt} = \omega$)

5.2.1. Uniforme circular motionحركة دائرية منتظمة

In this case the angular velocity ω is constant and therefore the linear velocity v is also constant, then $a_T = 0$.

The acceleration in this case is :$\vec{a} = \overrightarrow{a_N} = \frac{v^2}{R}\vec{n}$

5.2.2. Uniformly variable circular motion حركة دائرية متغيرة بانتظام

In this case the angular velocity ω is not constant and therefore the velocity v is not constant also, then $\vec{a} = a_T\vec{u} + a_N\vec{n}$.

The acceleration in this case is: $\vec{a} = \frac{dv}{dt}\vec{u} + \frac{v^2}{R}\vec{n}$ =$R\frac{d\omega}{dt}\vec{u} + R\omega^2\vec{n}$

5.3. Sinusoidal or harmonic motion حركة جيبية

The movement is called sinusoidal or harmonic if its evolution over time is written by the equation:

$$x(t) = A\sin(\omega t + \varphi)$$

A: amplitude, ω: angular frequency, and φ: phase.

$$\omega = \frac{2\pi}{T} = 2\pi f$$

T: period and f: frequency

- ***The speed***

$$v(t) = \frac{dx(t)}{dt} = A\omega\cos(\omega t + \varphi)$$

- ***The acceleration***

$$a(t) = \frac{dv(t)}{dt} = \frac{d^2x(t)}{dt^2} = -A\,\omega^2\sin(\omega t + \varphi) \Rightarrow a(t) = \frac{d^2x(t)}{dt^2} = -\,\omega^2\,x(t)$$

Note:
Another type of movement which is relative movement will be detailed in the next chapter.