University of Tlemcen **Academic year 2024-2025**
Faculty of Sciences **(L2 Maths)**
department of mathematics **Algebra 3 (Second Year)**

Worksheet N°2/ " Determinants "

Exercise 01:

(1) Calculate the following determinants:

$$\triangle_1 = \begin{vmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{vmatrix}, \triangle_2 = \begin{vmatrix} 3 & 16 & 24 & 33 \\ 1 & 5 & 7 & 9 \\ 5 & 27 & 36 & 55 \\ 7 & 38 & 51 & 78 \end{vmatrix}$$

$$\triangle_3 = \begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix} \text{ and } \triangle_4 = \begin{vmatrix} 1+a & a & a \\ b & 1+b & b \\ c & c & 1+c \end{vmatrix}.$$

(2) Show that $\triangle_5$ is zero without calculating it with:

$$\triangle_5 = \begin{vmatrix} 0 & 4 & 5 & 1 \\ 2 & 7 & 0 & 6 \\ 1 & 8 & 0 & 4 \\ 2 & 2 & 5 & 5 \end{vmatrix}.$$

(3) Show that $\triangle_6$ is divisible by 13 with:

$$\triangle_6 = \begin{vmatrix} 0 & 1 & 6 & 9 \\ 2 & 7 & 0 & 4 \\ 1 & 3 & 1 & 3 \\ 5 & 2 & 6 & 5 \end{vmatrix}.$$

Exercise 02:

(1) Show that for $n > 2$, the following determinant is equal to zero:

$$\triangle_1 = \begin{vmatrix} a_1-b_1 & a_1-b_2 & \cdots & a_1-b_n \\ a_2-b_1 & a_2-b_2 & \cdots & a_2-b_n \\ \vdots & \vdots & & \vdots \\ a_n-b_1 & a_n-b_2 & \cdots & a_n-b_n \end{vmatrix}. (a_i, b_i \in \mathbb{R})$$

(2) Calculate the determinant:

$$\triangle_n = \begin{vmatrix} 1+x^2 & x & 0 & \cdots & \cdots & \cdots & 0 \\ x & 1+x^2 & x & 0 & \cdots & \cdots & \cdots \\ 0 & x & 1+x^2 & x & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & x & 1+x^2 & x \\ 0 & 0 & \cdots & \cdots & 0 & x & 1+x^2 \end{vmatrix}.$$

(3) Calculate the determinant:

$$\delta_n = \begin{vmatrix} x & a_1 & a_2 & \cdots & a_{n-1} & 1 \\ a_1 & x & a_2 & \cdots & a_{n-1} & 1 \\ a_1 & a_2 & x & \cdots & a_{n-1} & 1 \\ \vdots & \vdots & a_3 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & & x & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_n & 1 \end{vmatrix}.$$

Exercise 03: Let:

$$\begin{array}{ccc} & \overset{k}{\leftrightarrow} \quad \overset{n-k}{\leftrightarrow} & \\ D = & \left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right) & \begin{array}{l} \}k \\ \}n-k \end{array} \end{array}$$

with $A \in M_k(\mathbb{k})$ and $C \in M_{n-k}(\mathbb{k})$. Show that: $\det D = \det A . \det C$.

Exercise 04: Let $A, B \in M_n(\mathbb{k})$, Show that:

(1)

$$\det \left(\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right) = \det(A+B)\det(A-B).$$

(2) If $AB = BA$:

$$\det \left(\begin{array}{c|c} A & -B \\ \hline B & A \end{array}\right) = \det(A^2+B^2).$$

Exercise 05: Let $\triangle_n$ the determinant defined by:

$$\triangle_n = \begin{vmatrix} a+b & b & \cdots & \cdots & b \\ a & a+b & b & \cdots & \vdots \\ \vdots & a & \ddots & & \vdots \\ \vdots & & & \ddots & b \\ a & \cdots & \cdots & a & a+b \end{vmatrix}, a \neq b, a, b \in \mathbb{R}.$$

(1) Show that: $\triangle_n = a\triangle_{n-1} + b^n$.

(2) Deduce by induction that : $\triangle_n = \frac{a^{n+1}-b^{n+1}}{a-b}$.

Exercise 06: In $\mathbb{R}^4$, let $v_1 = \begin{pmatrix} -1 \\ 2 \\ 3 \\ -4 \end{pmatrix}$, $v_2 = \begin{pmatrix} 3 \\ -2 \\ 5 \\ 1 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 6 \\ -3 \\ \alpha \\ \beta \end{pmatrix}$.

(1) Find α and β where $\{v_1, v_2, v_3\}$ are linearly independent.

(2) Find α and β where v_3 is spanned by v_1 and v_2.

Sincere wishes you success (MESSIRDI BACHIR)