University of Tlemcen **Academic year 2024-2025**
Faculty of Sciences **(L2 Maths)**
department of mathematics **Algebra 3 (Second Year)**

Worksheet N°1/ "Revision of linear algebra"

Exercise 01:

(1) Let $(\mathbb{R}^4, +, .)$ be the $\mathbb{R}-$ vector space and E_1 a sub space defined by:

$$E_1 = \{(x, y, z, t) \in \mathbb{R}^4 / 2x - y = 3z + t \text{ and } x - 4y + 5z - t = 0\}.$$

Determine the dimension of E_1.

(2) In $\mathbb{R}_4[X]$ the vector space of polynomials of degree less than or equal to 4 and with real coefficients. We put:

$$E_2 = \left\{P \in \mathbb{R}_4[X] / \int_0^1 P(x)dx = 0\right\}.$$

and

$$E_3 = \{P \in \mathbb{R}_4[X] / P(X) - P'(X)(X+2) = 0\}.$$

Find $\dim E_2$ and $\dim E_3$.

(3) Let $M_3(\mathbb{R})$ be the vector space of the matrices of order 3 and with real coefficients. Let E_4 the subspace of $M_3(\mathbb{R})$ defined by:

$$E_4 = \{A \in M_3(\mathbb{R}) / {}^tA = A\}.$$

Determine the dimension of E_4.

Exercise 02: (1) Do the following systems form E bases in each case?

(a) $S_1 = \{(2, 0, 1), (-4, 5, 6)\}$ with $E = \mathbb{R}^3$.

(b) $S_2 = \{(2, 4, 1), (-4, 5, 6), (-1, 2, 2)\}$ with $E = \mathbb{R}^3$.

(c) $S_3 = \{-2, 3 + X, X + X^2, 5X^2\}$ with $E = \mathbb{R}_2[X]$.

(d) $S_4 = \{-3, 3 + X, 5X + X^2\}$ with $E = \mathbb{R}_2[X]$.

(2) Are the following vectors linearly independent in $\mathbb{R}^4$?

$$v_1(1, 2, -1, 3), v_2(3, -2, 2, -1), v_3(-1, 4, -1, 2).$$

Determine a vector v_4 so that the family $\{v_1, v_2, v_3, v_4\}$is a basis of $\mathbb{R}^4$.

(3) Determine a basis and dimension of $F + G$ where:

$$F = span\{v_1, v_2\} \text{ and } G = span\{v_3, v_4\},$$

with $v_1 = (1, 0, 1, 1), v_2(2, 1, -1, 1), v_3(1, 1, -2, 0)$ and $v_4(0, 0, 0, 1)$.

Exercise 03: In $\mathbb{R}^4$ let the two subset:

$$E_1 = \left\{(a,b,c,d) \in \mathbb{R}^4 / b - 2c + d = 0\right\} \text{ and } E_2 = \left\{(a,b,c,d) \in \mathbb{R}^4 / a = d \text{ and } b = 2c\right\}.$$

(1) Determine a basis B_1 of E_1 and a basis B_2 of E_2.

(2) Find a basis of $E_1 \cap E_2$.

(3) deduce $E_1 + E_2$.

Exercise 04: Let E be a $\mathbb{k}$-vector space and f an endomorphism of E such that:

$$f \circ f = f.$$

Show that:

$$E = \ker f \oplus \operatorname{Im} f.$$

Exercise 05: Let E and F be two $\mathbb{k}$-vector spaces, $f \in \mathcal{L}(E,F)$ and $g \in \mathcal{L}(E,F)$ such that:

$$f \circ g \circ f = f \text{ and } g \circ f \circ g = g.$$

Show that:

$$E = \ker f \oplus \operatorname{Im} g.$$

Exercise 06: Let E be a finite-dimensional $\mathbb{k}$-vector space and f an endomorphism of E. Show that the following properties are equivalent:

$$(a) \ker f = \ker f^2; (b) \operatorname{Im} f = \operatorname{Im} f^2; (c) E = \ker f \oplus \operatorname{Im} f.$$

Exercise 07: The linear application defined by:

$$\begin{array}{cccc} f & : & \mathbb{R}^3 \to \mathbb{R}^2 \\ (x,y) & \mapsto & f(x,y,z) = (2x - 3y + z, 3x - y - 2z). \end{array}$$

Let be $B_2 = \{e_1, e_2, e_3\}$ and $B_3 = \{u_1, u_2\}$ the canonical basis of $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively.

(1) Find the matrix M associated with f relative to the basis B_2 and B_3.

(2) Let $C_2 = \{v_1, v_2, v_3\}$ is a basis $\mathbb{R}^2$, with $v_1 = (0,1,3)$, $v_2 = (-1,2,1)$ and $v_3 = (-2,-1,5)$. and $C_3 = \{w_1, w_2\}$ a basis of $\mathbb{R}^3$, with $w_1 = (1,3)$, $w_2 = (-1,2)$.

a) Find the change-of-coordinate matrix P from B_2 to C_2.

b) Find the change-of-coordinate matrix Q from B_3 to C_3.

c) If V has as components $\begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$ in B_2, determine the components of V in the basis C_2.

d) Deduct the matrix N associated with f relative to the basis C_2 and C_3.

Sincere wishes you success (MESSIRDI BACHIR)