Exercise Series No. 3

Exercise 1:

Compute the limits of the following functions:

$$1) \lim_{x \to 1^+} \left(\frac{1}{1-x} - \frac{1}{1-x^2} \right), \qquad 2) \lim_{x \to 1} \frac{x^n - 1}{x - 1}, n \in \mathbb{N}^* \qquad 3) \lim_{x \to +\infty} \frac{x - \sqrt{x}}{\ln(x) + x}$$

$$4) \lim_{x \to 0} \frac{\ln(1 + x^2)}{\sin^2(x)}, \qquad 5) \lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x} \text{ (SUPP)}, \qquad 6) \lim_{x \to +\infty} \sqrt{x + \sqrt{x}} - \sqrt{x} \text{ (SUPP)}$$

Exercise 2:

Study the continuity of the following functions:

$$1) f(x) = \begin{cases} x + \dfrac{\sqrt{x^2}}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \qquad 2)(\textbf{SUPP}) f(x) = \begin{cases} \dfrac{\sin(x)}{|x|} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Exercise 3:

Determine if the following functions are extendable by continuity at $x_0 = 0$.

$$1) f(x) = \frac{e^x - 1}{\sqrt{|x|}}, \qquad 2)(\textbf{SUPP}) f(x) = \begin{cases} \dfrac{1 - \cos(x)}{x^2} & \text{if } x > 0, \\ \dfrac{\sin(x)}{x} & \text{if } x < 0. \end{cases}$$

Exercise 4:

Study the differentiability on $\mathbb{R}$ of the following functions:

$$1) f(x) = x|x|, \qquad 2)(\textbf{SUPP}) f(x) = \begin{cases} \dfrac{|x|\sqrt{x^2 - 2x + 1}}{x - 1} & \text{if } x \neq 1, \\ 1 & \text{if } x = 1. \end{cases}$$

Exercise 5:

1) Using the Mean Value Theorem, show that

$$\frac{1}{1+x} < \ln(1+x) - \ln(x) < \frac{1}{x}.$$

2) Compute

$$\lim_{x \to +\infty} x(\ln(1+x) - \ln(x))$$

and deduce

$$\lim_{x \to +\infty} \left(1 + \frac{1}{x} \right)^x.$$