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Worksheet N°1
The real numbers
LMD 1st year 2024-2025

Exercise 1

1. Let $a \in \mathbb{R}$, Show that, if a^2 is factor of 2 , then a is also factor of 2.
2. Show that, the numbers $\log_2(3)$ and $\sqrt{3}$ are irrationals numbers.
3. We suppose that π is irrational, show that, the number $\left(\frac{3}{\pi}\right)$ is irrational.
4. Show that $\forall (p,q) \in \mathbb{N},\, p \neq q : (\sqrt{p} + \sqrt{q}) \notin \mathbb{Q} \Rightarrow (\sqrt{p} - \sqrt{q}) \notin \mathbb{Q}$.
5. **(optional)**.We suppose that $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{6}$ are irrationals. Show that, the numbers $\sqrt{2}, \sqrt{3}, \sqrt{2} + \sqrt{3}$ are irrationals numbers.
6. **(optional)** We suppose that $\sqrt{2}$ is irrational.
   (a) Show that $a = 6 + 4\sqrt{2}$ and $b = 6 - 4\sqrt{2}$ are irrationals numbers.
   (b) Calculat $\sqrt{a \times b}$, then show that $(\sqrt{a} + \sqrt{b})$ is an rational.

Exercise 2

1. Resolve in $\mathbb{R}$:$|x-1| + |x+1| = 4$.
2. let x and y two real numbers, show that
   (a) $|x| - |y| \leq |x - y|$.
   (b) $|x| + |y| \leq |x + y| + |x - y|$
3. **(optional)**
   (a) $1 + |xy - 1| \leq (1 + |x - 1|)(1 + |y - 1|)$.
   (b) For all $x \in \mathbb{R}$, we put $f(x) = \dfrac{|x|}{1+|x|}$.
   Show that, $\forall x, y \in \mathbb{R},\, f(x+y) \leq f(x) + f(y)$.

Exercise 3

1. Resolve in $\mathbb{R}$

$$\begin{array}{rclll} E(\frac{2x+1}{3}) - 2 & = & 0, & E(x+a) = 2, a \in \mathbb{R}. & E(x^2 - x + 2) - x = -1, \\ E(x) & \geq & 1, & -1 \leq E(3x) \leq 1, & E(x) + |x-1| = x, \end{array}$$

2. Show that $\forall n \in \mathbb{N}^*$, we have $E(x+n) = E(x) + n$ and $E(\frac{1}{n}E(nx)) = E(x)$.

3. **(optional)** Let $x, y \in \mathbb{R}$, Show that

$$E(x) + E(y) \leq E(x+y)$$

and

$$E(x+y) \leq E(x) + E(y) + 1.$$

Exercise 4

1. Give the definition of interval $I \subset \mathbb{R}$.

2. Let A, B, C the subsets of $\mathbb{R}$, such that
$A = \{x \in \mathbb{R},\ x^2 < 1\}$, $B = \{x \in \mathbb{R}, (x-3)(x+2) \geq 0\} \cap [-4, 4]$,
$C = \{x \in \mathbb{R}^*,\ \dfrac{1}{x} > 2\}$, $D = \{x \in \mathbb{R},\ -1 \leq E(3x) \leq 1\} \cup [1, \pi[$.

   (a) Put these sets in the form of an interval of $\mathbb{R}$, or an interval union.

   (b) Why is the set C not an interval?

   (c) Find the set of all upper bound, lower bound, supremum, infimum, maximum and minimum if there exists.

Exercise 5

1. Find the sets of all upper bound, lower bound, supremum, infimum, maximum and minimum if there exists of:
$A = \left\{\frac{x+1}{x+2},\ x \in \mathbb{R}, x \leq -3\right\}$, $B = \left\{\frac{n+3}{n+2},\ n \in \mathbb{N}\right\}$, $C = \left\{(-1)^n + \frac{1}{n^2},\ n \in \mathbb{N}^*\right\}$.
$D = \left\{2 - \frac{8}{n+4},\ n \in \mathbb{N}\right\}$, $E = \left\{\frac{(-1)^n}{n+1} + \frac{(-1)^n + 2}{3}\ ,\ n \in \mathbb{N}\right\}$.
(D and E optional).

Exercise 6
Let A and B be nonempty subsets of real numbers, prove that

1. If $(A \subset B) \Rightarrow (\sup A \leq \sup B)$ et $(\inf B \leq \inf A)$.

2. $\inf(A \cup B) = \min\{\inf A,\ \inf B\}$.

3. $\sup(A \cup B) = \max\{\sup A,\ \sup B\}$.**(optional)**