Algèbre 1
Tutorials series N° 2
Sets and applications

Sets

Exercice 1.
Let A, B, C be subsets of a set E. Show that :

1. $A \cup B = B \Longleftrightarrow A \subset B$.
2. $A \cap B = A \Longleftrightarrow A \subset B$.
3. $A \cup B = A \cap C \Longleftrightarrow B \subset A \subset C$.
4. $\begin{cases} A \cap B = A \cap C \\ A \cup B = A \cup C \end{cases} \Longleftrightarrow B = C$.
5. $\complement_E(A \cup B) = \complement_E A \cap \complement_E B$ *and* $\complement_E(A \cap B) = \complement_E A \cup \complement_E B$ *(De Morgan's laws).*

Exercice 2. *Let E be the following set :*

$$E = \{-5, -1.1, \pi, 10\}$$

1. *Determine the cardinal of the* set of parts *of the set E, given by card$\mathcal{P}(E)$.*
2. *Determine $\mathcal{P}(E)$.*
3. *Among the following symbols and notations :*

$$\in, \notin, \subset, \not\subset, \cap, \cup, \varnothing, E, =$$

put the appropriate one in the following relationships : :

$$-3 \ldots E,\ -1.1 \ldots \mathcal{P}(E),\ \{\pi\} \ldots E,\ \{10\} \ldots \mathcal{P}(E),\ \varnothing \ldots E,\ \varnothing \ldots \mathcal{P}(E),\ \{-5, \pi, 10\} \ldots \mathcal{P}(E)$$
$$\{-3, -1.1\} \ldots \{-1.1, \pi\} \ldots \{-1.1\},\ \{\{\pi\}\} \ldots \mathcal{P}(E),\ \{-3, -1.1\} \ldots E,\ \{\varnothing\} \ldots \mathcal{P}(E)$$

Applications

Exercice 3. *Let f be the following application :*

$$f :]-\sqrt{3}, +\infty[\to \mathbb{R}_+$$
$$x \mapsto f(x) = x^2 - 3$$

1. *Determine the direct image of the set $A = \{-\sqrt{2}, 3, 0.5, \pi, 8\}$ by the application f.*
2. *Determine the reciprocal image of the set $B = [0, 1]$ by the application f.*

Exercice 4. *Check whether the following applications are injective, surjective and bijective :*

1. $f : \mathbb{R} \longrightarrow \mathbb{R}_+$ *defined by :* $\forall x \in \mathbb{R}, f(x) = |x|$.
2. $h : \mathbb{R}_+ \longrightarrow \mathbb{R}$ *defined by :* $\forall x \in \mathbb{R}_+, h(x) = x^2$.
3. $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *defined by :* $\forall (x, y) \in \mathbb{R}^2, g(x, y) = (x, xy)$.
4. $k : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *defined by :* $\forall x \in \mathbb{R}_+, k(x) = \sqrt{x}$.

Determine the expression of the application $h \circ k$, its starting set, its arrival set and deduce that it is injective.

Exercice 5. *Let f be defined by :*

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \mapsto f(x) = \frac{4x}{x^2 + 1}$$

1. *Check that for any non-zero real a, we have $f(a) = f(\frac{1}{a})$. Is f injective ?*
2. *Let h be the application defined by :*

$$h : [1, +\infty[\longrightarrow \mathbb{R}$$
$$x \mapsto h(x) = f(x)$$

   (a) *Show that h is injective.*
   (b) *Check that :* $\forall x \in [1, +\infty[, h(x) \leq 2$.
   (c) *Show that h is a bijection from $[1, +\infty[$ to $]0, 2]$ and find h^{-1}.*