Algebra 1
Tutorials series N° 1
Logic Concepts

Exercise 1.
Let P, Q, R be three statements. Establish the truth tables of the following statements and deduce whether they are equivalent :

1. $P \vee (Q \wedge R)$ *and* $(P \vee Q) \wedge (P \vee R)$
2. $P \implies Q$ *and* $\overline{P} \vee Q$
3. $P \implies Q$ *and* $Q \implies P$

Exercise 2. *Let P and Q be two statements.*
Show the following equivalences :

1. $\overline{P \implies Q} \Leftrightarrow P \wedge \overline{Q}$
2. $P \implies Q \Leftrightarrow \overline{Q} \implies \overline{P}$
3. $(P \Leftrightarrow Q) \Leftrightarrow (P \implies Q) \wedge (Q \implies P)$
4. $P \oplus Q \Leftrightarrow (P \wedge \overline{Q}) \vee (\overline{P} \wedge Q)$
5. $(P \oplus Q) \oplus Q \Leftrightarrow P$

Are the following statements Tautologies ? :

1. $P \vee \overline{P}$
2. $P \wedge \overline{P}$

Exercise 3. *Let P et Q be two statements. Give the negations of the following statements : :*

1. $P \wedge Q$
2. $[(P \wedge Q) \vee R] \implies (P \wedge R)$
3. $P \Leftrightarrow Q$
4. $P \oplus Q$

Exercise 4.

1. *Is the statement :* $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x^2 + y < 0,$ *true ?*
   Swap the quantifiers and deduce the truth value of the resulting statement.
2. *Give the negation of the following statement, and say whether it is true or false :*
   $\forall y \in \mathbb{R}, \exists n \in \mathbb{N}, y \leq n.$
   Deduce the truth value of the given statement.

Exercise 5.

1. *Let $a \in \mathbb{N}$. Study according to the values of a the existence or not of solutions for the equation* $x^2 + 2ax + 3 = 0$.
2. *Show by contraposition that :* $\forall p \in \mathbb{N}, \quad p^2$ *is even* $\implies p$ *is even.*
3. *Use reasoning by contradiction to show the previous statement (supp).*
4. *Show by contradiction that $\sqrt{2}$ is irrational, that is to say $\sqrt{2} \notin \mathbb{Q}$.*
5. *Are the following two statements true :*

$$\forall x \in]-\infty, 3[, \; x^2 < 9,$$
$$\forall x, y \in \mathbb{R}, \; x^2 + y^2 \geq xy.$$

6. *Show that for any integer $n \geq 4$, $n! \geq 2^n$.*
7. *Show that :* $\forall n \in \mathbb{N}^*, \sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$. *It is given that* $\sum_{k=1}^{n} k = 1 + 2 + 3 + 4 + \cdots + n$.
8. *Shwo that :* $\forall n \in \mathbb{N}, 7^n - 1$ *is divisible by* 6.
9. *Show that :* $\forall n \in \mathbb{N}, 4^n + 6n - 1$ *is a multiple of* 9 *(supp).*