Solution Exercise 1.

1. (a) $\forall x \in \mathbb{Z},\ x - x = 0$, *then* $(x - x)$ *is a multiple of* 2 *and* 3, *hence* $\mathcal{R}$ *is reflexive.*

   (b) *Let* $x, y \in \mathbb{Z}$ *such that* $x\mathcal{R}y$.

$$\begin{aligned} x\mathcal{R}y &\Longrightarrow (x-y) \text{ is a multiple of } 2 \text{ and } 3.\\ &\Longrightarrow -(x-y) \text{ is a multiple of } 2 \text{ and } 3.\\ &\Longrightarrow (y-x) \text{ is a multiple of } 2 \text{ and } 3.\\ &\Longrightarrow y\mathcal{R}x \end{aligned}$$

   Then $\mathcal{R}$ *is symmetrical.*

   (c) *Let* $x, y, z \in \mathbb{Z}$ *such that* $x\mathcal{R}y$ *and* $y\mathcal{R}z$.

$$x\mathcal{R}y \text{ and } y\mathcal{R}z \Longrightarrow \exists k, k^{'}, l, l^{'} \in \mathbb{Z}, (x-y = 2k \text{ and } x-y = 3k^{'}) \text{ and } (y-z = 2l \text{ and } y-z = 3l^{'})$$

   then,

$$x-z = (x-y)+(y-z) = 2k+2l = 2(k+l) \text{ and } x-z = (x-y)+(y-z) = 3k^{'}+3l^{'} = 3(k^{'}+l^{'})$$

   hence $x\mathcal{R}z$, *i.e.* $\mathcal{R}$ *is transitive.*

2. (a) $\forall x \in \mathbb{Z},\ x - x = 0$, *then* $(x - x)$ *is a multiple of* 2 *or* 3, *hence* $\mathcal{P}$ *is reflexive.*

   (b) *Let* $x, y \in \mathbb{Z}$ *such that* $x\mathcal{P}y$.

$$\begin{aligned} x\mathcal{R}y &\Longrightarrow (x-y) \text{ is a multiple of } 2 \text{ or } 3.\\ &\Longrightarrow -(x-y) \text{ is a multiple of } 2 \text{ or } 3.\\ &\Longrightarrow (y-x) \text{ is a multiple of } 2 \text{ or } 3.\\ &\Longrightarrow y\mathcal{R}x \end{aligned}$$

   Then $\mathcal{P}$ *is symmetrical.*

   (c) *Let* $2, 4, 7 \in \mathbb{Z}$, *then*

   — $2\mathcal{P}4$ *because* $2 - 4 = -2$ *is a multiple of* 2 *despite it is not a multiple of* 3, *but the definition of* $\mathcal{P}$ *gives the possibiity for* $2 - 4$ *to be a multiple of* 2 ***or*** *a multiple of* 3.

   — $4\mathcal{P}7$ *because* $4 - 7 = -3$ *is a multiple of* 3 *despite it is not a multiple of* 2, *but the definition of* $\mathcal{P}$ *gives the possibiity for* $4 - 7$ *to be a multiple of* 2 ***or*** *a multiple of* 3.

   — *But* $2\not{\mathcal{P}}7$ *because* $2 - 7 = 5$ *is neither à multiple of* 2 *nor a multiple of* 3.

   Then $\mathcal{P}$ *is not transitive.*

3. (a) $\forall (a, a^{'}) \in \mathbb{N} \times \mathbb{N},\ a + a^{'} = a + a^{'} \Longrightarrow (a, a^{'})\mathcal{S}(a, a^{'})$, *then* $\mathcal{S}$ *is reflexive.*

   (b) *Let* $(a, a^{'}), (b, b^{'}) \in \mathbb{N} \times \mathbb{N}$ *such that* $(a, a^{'})\mathcal{S}(b, b^{'})$.

$$\begin{aligned} (a, a^{'})\mathcal{S}(b, b^{'}) &\Longrightarrow a + a^{'} = b + b^{'}\\ &\Longrightarrow b + b^{'} = a + a^{'}\\ &\Longrightarrow (b, b^{'})\mathcal{S}(a, a^{'}) \end{aligned}$$

   Hence $\mathcal{S}$ *is symmetrical.*

   (c) *Let* $(a, a^{'}), (b, b^{'}), (c, c^{'}) \in \mathbb{N} \times \mathbb{N}$ *such that* $(a, a^{'})\mathcal{S}(b, b^{'})$ *and* $(b, b^{'})\mathcal{S}(c, c^{'})$.

$$\begin{aligned} (a, a^{'})\mathcal{S}(b, b^{'}) \text{ and } (b, b^{'})\mathcal{S}(c, c^{'}) &\Longrightarrow a + a^{'} = b + b^{'} \text{ and } b + b^{'} = c + c^{'}\\ &\Longrightarrow a + a^{'} = c + c^{'}\\ &\Longrightarrow (a, a^{'})\mathcal{S}(c, c^{'}) \end{aligned}$$

   Then $\mathcal{S}$ *is transitive.*

The relations $\mathcal{R}$ and $\mathcal{S}$ are equivalence relations.

Solution Exercise 2.

1. *Let's show that $\mathcal{P}$ is reflexive. Let $x \in \mathbb{R}$, then $\cos^2 x + \sin^2 x = 1$ hence, $x\mathcal{P}x$, i.e. $\mathcal{P}$ is reflexive.*
2. *Let $x, y \in \mathbb{R}$ such that $x\mathcal{P}y$.*

$$\begin{aligned} x\mathcal{P}y &\Longrightarrow \cos^2 x + \sin^2 y = 1 \\ &\Longrightarrow 1 - \sin^2 x + 1 - \cos^2 y = 1 \\ &\Longrightarrow \cos^2 y + \sin^2 x = 1 \\ &\Longrightarrow y\mathcal{P}x \end{aligned}$$

then $\mathcal{P}$ is symmetrical.

3. *Let $x, y, z \in \mathbb{R}$ such that $x\mathcal{P}y$ and $y\mathcal{P}z$.*

$$\begin{aligned} x\mathcal{P}y \text{ and } y\mathcal{P}z &\Longrightarrow \begin{cases} \cos^2 x + \sin^2 y = 1 \\ \cos^2 y + \sin^2 z = 1 \end{cases} \\ &\Longrightarrow \cos^2 x + \sin^2 y + \cos^2 y + \sin^2 z = 2 \\ &\Longrightarrow \cos^2 x + \sin^2 z = 1 \\ &\Longrightarrow x\mathcal{P}z \end{aligned}$$

then $\mathcal{P}$ is transitive.

We conclude that $\mathcal{P}$ is an equivalence relation.

Solution Exercise 3.

1. (a) *Let $x \in \mathbb{Z}$, then $x - x = 0 = 7 \cdot 0$, hence $x\mathcal{R}x$. $\mathcal{R}$ is reflexive.*

   (b) *Let $x, y \in \mathbb{Z}$ such that $x\mathcal{R}y$.*

$$\begin{aligned} x\mathcal{R}y &\Longrightarrow \exists m_i n\mathbb{Z},\ x - y = 7m \\ &\Longrightarrow y - x = 7m^{'} \text{ with } m^{'} = -m \in \mathbb{Z} \\ &\Longrightarrow y\mathcal{R}x \\ &\Longrightarrow \mathcal{R} \text{ is symmetrical.} \end{aligned}$$

   (c) *Let $x, xy, z \in \mathbb{Z}$ such that, $x\mathcal{R}y$ and $y\mathcal{R}z$.*

$$\begin{aligned} x\mathcal{R}y \text{ and } y\mathcal{R}z &\Longrightarrow \exists m, m^{'} \in \mathbb{Z},\ x - y = 7m \text{ and } y - z = 7m^{'} \\ &\Longrightarrow x - z = 7m^{''},\ \text{with } m^{''} = m + m^{'} \in \mathbb{Z} \\ &\Longrightarrow x\mathcal{R}z \\ &\Longrightarrow \mathcal{R} \text{ is transitive.} \end{aligned}$$

Then $\mathcal{R}$ is an equivalence relation on $\mathbb{Z}$.

2. *Let $x \in \mathbb{Z}$ then $\overline{x} = \{y \in \mathbb{Z}, y\mathcal{R}x\}$.*

$$\begin{aligned} y\mathcal{R}x &\Longleftrightarrow \exists m \in \mathbb{Z}, y - x = 7m \\ &\Longleftrightarrow y = 7m + x \end{aligned}$$

This means that the equivalence class of $x \in \mathbb{Z}$ is the set of the integers y whose remainder of its Euclidian division by 7 is x. But in the Euclidiant division of an integer y by 7, the remainder can be one of the following : $0, 1, 2, 3, 4, 5, 6$ and no one more. That is, any integer divided by 7 has one of the previous remainder. So some[1] *integers will have as remainder 0, other ones will have 1 as remainder and so on till the last integers who have 6 as reainder, and hence no more intergers are left. So the equivalence classes are : $\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}$*

1. I wrote *some* but the reality is that there are infinite integers who have the same remainder.

3. *The quotient set is :*

$$\mathbb{Z}/\mathcal{R} = \mathbb{Z}/7\mathbb{Z} = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}\}$$

Solution Exercise 4.

1. (a) *Let* $p \in \mathbb{N}^*$ *then* $p^1 = p$, *hence* $\exists n = 1 \in \mathbb{N}^*, p^n = p$ *i.e.* $p\mathcal{T}p$, *that is* $\mathcal{T}$ *is reflexive.*

   (b) *Let* $p, q \in \mathbb{N}^*$ *such that* $p\mathcal{T}q$ *and* $q\mathcal{T}p$.

$$\begin{aligned} p\mathcal{T}q \text{ and } q\mathcal{T}p &\Longrightarrow \exists n, m \in \mathbb{N}^*,\ p^n = q \text{ and } q^m = p \\ &\Longrightarrow (q^n)^m = q \\ &\Longrightarrow q^{nm} = q \\ &\Longrightarrow n = m = 1 \text{ because } n, m \in \mathbb{N}^* \\ &\Longrightarrow p = q \\ &\Longrightarrow \mathcal{T} \text{ is antisymmetric.} \end{aligned}$$

   (c) *Let* $p, q, r \in \mathbb{N}^*$ *such that* $p\mathcal{T}q$ *and* $q\mathcal{T}r$.

$$\begin{aligned} p\mathcal{T}q \text{ and } q\mathcal{T}r &\Longrightarrow \exists n, m \in \mathbb{N}^*,\ p^n = q \text{ and } q^m = r \\ &\Longrightarrow (p^n)^m = r \\ &\Longrightarrow \exists l \in \mathbb{N}^*,\ p^l = r \text{ with } l = nm \in \mathbb{N}^* \\ &\Longrightarrow \mathcal{T} \text{ is transitive.} \end{aligned}$$

   Then $\mathcal{T}$ *is an order relation.*

2. $\mathcal{T}$ *is a partial order. Take for example* $p = 3$ *and* $q = 4$, $p, q \in \mathbb{N}^*$, *then* $\forall n \in \mathbb{N}^*,\ p^n \neq q$ *and* $\forall n \in \mathbb{N}^*,\ q^n \neq p$ *that is* $p \not\mathcal{T} q$ *and* $q \not\mathcal{T} p$.