Solution Exercise 1.

1. $P \vee (Q \wedge R)$ *and* $(P \vee Q) \wedge (P \vee R)$

P	Q	R	$Q \wedge R$	$P \vee (Q \wedge R)$	$P \vee Q$	$P \vee R$	$(P \vee Q) \wedge (P \vee R)$
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0
				①			②

The two columns ① and ② are identical, then the two statements are equivalent.

2. $P \implies Q$ *and* $\overline{P} \vee Q$

P	Q	$P \implies Q$	$\overline{P}$	$\overline{P} \vee Q$
1	1	1	0	1
1	0	0	0	0
0	1	1	1	1
0	0	1	1	1
		①		②

The two columns ① and ② are identical, then the two statements are equivalent.

3. $P \implies Q$ *and* $Q \implies P$

P	Q	$P \implies Q$	$Q \implies P$
1	1	1	1
1	0	0	1
0	1	1	0
0	0	1	1
		①	②

The two columns ① and ② are not identical, then there is no equivalence between the two statements.

Solution Exercise 2. *Let's show the equivalences :*

1. $\overline{P \implies Q} \equiv$ [1] $P \wedge \overline{Q}$

P	Q	$\overline{Q}$	$P \implies Q$	$\overline{P \implies Q}$	$P \wedge \overline{Q}$
1	1	0	1	0	0
1	0	1	0	1	1
0	1	0	1	0	0
0	0	1	1	0	0
				①	②

The two columns ① and ② are identical, then the two statements are equivalent.

1. $\equiv$ is another notation for the equivalence.

2. $P \implies Q \equiv \overline{Q} \implies \overline{P}$

P	Q	$\overline{P}$	$\overline{Q}$	$P \Longrightarrow Q$	$\overline{Q} \Longrightarrow \overline{P}$
1	1	0	0	1	1
1	0	0	1	0	0
0	1	1	0	1	1
0	0	1	1	1	1
				①	②

The two columns ① and ② are identical, then the two statements are equivalent.

3. $(P \Leftrightarrow Q) \equiv (P \implies Q) \wedge (Q \implies P)$

P	Q	$P \Longleftrightarrow Q$	$P \Longrightarrow Q$	$Q \Longrightarrow P$	$(P \Longrightarrow Q) \wedge (Q \Longrightarrow P)$
1	1	1	1	1	1
1	0	0	0	1	0
0	1	0	1	0	0
0	0	1	1	1	1
		①			②

The two columns ① and ② are identical, then the two statements are equivalent.

4. $P \oplus Q \equiv (P \wedge \overline{Q}) \vee (\overline{P} \wedge Q)$

P	Q	$\overline{P}$	$\overline{Q}$	$P \oplus Q$	$P \wedge \overline{Q}$	$\overline{P} \wedge Q$	$(P \wedge \overline{Q}) \vee (\overline{P} \wedge Q)$
1	1	0	0	0	0	0	0
1	0	0	1	1	1	0	1
0	1	1	0	1	0	1	1
0	0	1	1	0	0	0	0
				①			②

The two columns ① and ② are identical, then the two statements are equivalent.

5. $(P \oplus Q) \oplus Q \equiv P$

P	Q	$P \oplus Q$	$(P \oplus Q) \oplus Q$
1	1	0	1
1	0	1	1
0	1	1	0
0	0	0	0
①			②

The two columns ① and ② are identical, then the two statements are equivalent.

Let's now verify if the given statments are tautologies

1. $P \vee \overline{P}$

P	$\overline{P}$	$P \vee \overline{P}$
1	0	1
0	1	1

The last column contains just ones, hence the statement is a tautology.

2. $P \wedge \overline{P}$

P	$\overline{P}$	$P \wedge \overline{P}$
1	0	0
0	1	0

The last column contains just zeros, then the statement is not a tautology; it's rather a contradiction.

Solution Exercise 3.

1.

$$\overline{P \wedge Q} \equiv \overline{P} \vee \overline{Q}$$

2.

$$\begin{aligned}\overline{[(P \wedge Q) \vee R] \implies (P \wedge R)} &\equiv [(P \wedge Q) \vee R] \wedge (\overline{P \wedge R}) \\ &\equiv [(P \wedge Q) \vee R] \wedge (\overline{P} \vee \overline{R})\end{aligned}$$

3.

$$\begin{aligned}\overline{P \Leftrightarrow Q} &\equiv \overline{(P \implies Q) \wedge (Q \implies P)} \\ &\equiv \overline{(P \implies Q)} \vee \overline{(Q \implies P)} \\ &\equiv (P \wedge \overline{Q}) \vee (Q \wedge \overline{P})\end{aligned}$$

This means that the negation of : P and Q are equivalent*, is :* one of the statements is true and the other is false*.*

4.

$$\begin{aligned}\overline{P \oplus Q} &\equiv \overline{(P \wedge \overline{Q}) \vee (Q \wedge \overline{P})} \\ &\equiv (\overline{P} \vee Q) \wedge (\overline{Q} \vee P) \\ &\equiv (P \implies Q) \wedge (Q \implies P) \\ &\equiv P \iff Q\end{aligned}$$

Solution Exercise 4.

1. *Is the statement :* $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x^2 + y < 0$*, true ?*
It's clear that for any $x \in \mathbb{R},\ x^2 \geq 0$*. Is there a* $y \in \mathbb{R}$ *such that* $x^2 + y < 0$ *? Let's consider* y *as the unknown of the inequation and let's try to solve it :*

$$x^2 + y < 0 \implies y < -x^2$$

and that's it, every $y \in \mathbb{R}$ *such that* $y < -x^2$*, and not just an only one, will do the job i.e.* $x^2 + y < 0$*. Hence this statement is true.*
Now let's swap the quantifiers :
• *We obtain the following statement :*

$$\exists y \in \mathbb{R},\ \forall x \in \mathbb{R},\ x^2 + y < 0$$

For $y \geq 0$ *the inequality* $x^2 + y < 0$ *is obviously false for any* $x \in \mathbb{R}$*.*
So for the inequality to hold, it's necessary that $y < 0$*, but in this case :*

$$x^2 + y < 0 \implies -\sqrt{-y} < x < \sqrt{y}$$

hence for $x \in]-\infty, -\sqrt{-y}] \cup [\sqrt{y}, +\infty[$ *the inequality doesn't hold.*
Conclusion : the statement is false.
• *Another way to make the swap is the following statement :*

$$\exists x \in \mathbb{R}, \forall y \in \mathbb{R},\ x^2 + y < 0$$

This statement is obviously false, because for any $x \in \mathbb{R}$*, if* $y = -x^2$ *then we obtain* $x^2 + y = 0$ *i.e. the inequality is not verified. Hence this statement is also false.*

2. *Let's write the negation of the statement :* $\forall y \in \mathbb{R}, \exists n \in \mathbb{N}, y \leq n$ *:*

$$\overline{\forall y \in \mathbb{R}, \exists n \in \mathbb{N}, y \leq n} \Longleftrightarrow \exists y \in \mathbb{R}, \forall n \in \mathbb{N}, y > n$$

This statement is false, because for $y \in \mathbb{R}_-$*, then* $y \leq 0$ *which contradicts the statement and if* $y \in \mathbb{R}_+$*, then if we take* $n = [y] + 1$*, we obtain* $y < n$ *which is a contradiction too with the statement.* $[y]$ *is called the* integer part*, it is the greatest integer less than or equal to* y*. We deduce that the original statement is true.*

Solution Exercise 5.

1. $x^2 + 2ax + 3 = 0$ *is a second degree equation of variable* x*. We will use a direct reasoning to study this equation. The discriminant of the equation is then* $\Delta = 4a^2 - 12$*, so :*
$\Delta > 0 \Longleftrightarrow \{a < -\sqrt{3}$ *or* $\sqrt{3} < a\}$*,* $\Delta = 0 \Longleftrightarrow a = \pm\sqrt{3}$ *and* $\Delta < 0 \Longleftrightarrow \{-\sqrt{3} < a < \sqrt{3}\}$*.*
Hence :
   (a) *If* $a \in]-\infty, -\sqrt{3}[\cup]\sqrt{3}, +\infty[$ *then the equation has two solutions :*

$$x_1 = \frac{-2a - \sqrt{\Delta}}{2} \qquad x_2 = \frac{-2a + \sqrt{\Delta}}{2}$$

   (b) *If* $a = \pm\sqrt{3}$*, the equation has a unique solution :*

$$x = \frac{-2a}{2} = -a \text{ hence } x = \sqrt{3} \text{ if } a = -\sqrt{3} \text{ and } x = -\sqrt{3} \text{ if } a = \sqrt{3}$$

   (c) *If* $a \in]-\sqrt{3}, \sqrt{3}[$ *then the equation has no solutions.*

   If we suppose $a \in \mathbb{N}$*, then* $\Delta \neq 0$*, hence there is just two cases :*
   (a) *If* $a \in \{2, 3, 4, \dots\}$ *then the equation has two solutions :*

$$x_1 = \frac{-2a - \sqrt{\Delta}}{2} \qquad x_2 = \frac{-2a + \sqrt{\Delta}}{2}$$

   (b) *If* $a \in \{0, 1\}$ *then the equation has no solutions.*

2. *We have to show by contraposition that :* $\forall p \in \mathbb{N}$*,* p^2 *is even* $\Longrightarrow$ p *is even.*
We have the equivalence :

$$(p^2 \text{ is even} \Longrightarrow p \text{ is even}) \Longleftrightarrow (p \text{ is odd} \Longrightarrow p^2 \text{ is odd})$$

Let's show the second implication which is easier to do :
If $p \in \mathbb{N}$ *such that* p *is odd, that is* $\exists k \in \mathbb{N},\ p = 2k + 1$*, then* $p^2 = (2k+1)^2 = 2k^{'} + 1$ *with* $k^{'} = 2k^2 + 2k \in \mathbb{N}$*, which means that* p^2 *is odd, and here we'r finished.*

3. *Let's try to prove by contradiction the same statement. (supplementary question)*
Suppose that : $\overline{\forall p \in \mathbb{N},\ p^2 \text{ is even} \Longrightarrow p \text{ is even}}$*, is true, that is* $\exists p \in \mathbb{N}$*,* p^2 *is even* ***and*** p *is odd.*

$$\begin{aligned} p \text{ is odd} &\Longrightarrow \exists k \in \mathbb{N}, p = 2k + 1 \\ &\Longrightarrow p^2 = 2k^{'} + 1 \\ &\Longleftrightarrow p^2 \text{ is odd, this is a contradiction with the hypothesis which says that } p^2 \text{ is even.} \end{aligned}$$

4. *Now we show by contradiction that* $\sqrt{2} \notin \mathbb{Q}$*.*
Soppose that $\sqrt{2} \in \mathbb{Q}$*, hence there exists* $p \in \mathbb{N}$ *and* $q \in \mathbb{N}^*$ *such that* $\sqrt{2} = \frac{p}{q}$ *with* $\frac{p}{q}$ *being reduced i.e. 1 is the only common divisor for* p *and* q*.*
$\sqrt{2} = \frac{p}{q}$ *implies that* $p^2 = 2q^2$*, this means that* p^2 *is even and hence* p *is even too (take a look*

at the question 2).
p is even is equivalent to say that there exists a $k \in \mathbb{N}$ *such that* $p = 2k$.
We obtain, then, from $p^2 = 2q^2$ *that* $4k^2 = 2q^2$ *and hence* $q^2 = 2k^2$ *and like p, we obtain that q is even too, which means that 2 is a common divisor of p and q and this is a contradiction because the fraction* $\frac{p}{q}$ *is reduced.*

5. *This statement* $\forall x \in]-\infty, 3[,\ x^2 < 9$ *is false, we can prove it by a counter example : Let* $x = -4$, *then* $x \in]-\infty, 3[$ *but* $x^2 = 16 > 9$.
The statement $\forall x, y \in \mathbb{R},\ x^2 + y^2 \geq xy$ *is true, we will prove it by a direct reasoning.*
Let $x, y \in \mathbb{R}$, *then* $(x-y)^2 \geq 0$ *hence* $x^2 + y^2 - 2xy \geq 0$ *which leads to* $x^2 + y^2 \geq 2xy$, *here we have two cases :*

case 1 : $xy < 0$ *i.e. x and y are of opposite signs, then* $x^2 + y^2 \geq xy$ *is obvious since* $x^2 + y^2 > 0$ *and* $xy < 0$.

case 2 : $xy \geq 0$ *i.e. x and y are both positive or both negative, then if we multiply the inequality* $2 > 1$ *by* xy, *its direction doesn't change and we obtain* $2xy > xy$. *Hence* $x^2 + y^2 \geq xy$. *(The null case i.e.* $x = 0$ *or* $y = 0$ *or* $x = y = 0$ *is included in this case 2).*

6. *We show by induction that for any integer* $n \geq 4$, $n! \geq 2^n$.
1.***Initialization :*** *For* $n = 4$ *we have :*

$$4! = 24 \geq 2^4 = 16$$

2.***Heredity :*** *Suppose that for a given* $n \geq 4$ *we have* $n! \geq 2^n$. *Using this hypothesis, let's show that* $(n+1)! \geq 2^{n+1}$.

$$\begin{aligned}(n+1)! &= (n+1) \cdot n! \\ &\geq (n+1) \cdot 2^n \text{ because of the induction hypothesis} \\ &\geq 2 \cdot 2^n \text{ because } \forall n \geq 4,\ n+1 > 2 \\ &= 2^{n+1}\end{aligned}$$

3.***Conclusion :*** $\forall n \geq 4,\ n! \geq 2^n$.

7. *Let's prove by induction the statement :* $\forall n \in \mathbb{N}^*, \sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$. *with* $\sum_{k=1}^{n} k = 1 + 2 + 3 + 4 + \cdots + n$. *In order to lighten the writing let's put :*

$$A(n) = \sum_{k=1}^{n} k \quad \text{and} \quad B(n) = \frac{1}{2}n(n+1)$$

1.***Initialization :*** *For* $n = 1$, *we have* $A(1) = B(1)$, *that is :*

$$\sum_{k=1}^{1} k = \frac{1}{2} \cdot 1 \cdot (1+1)$$

because : $\sum_{k=1}^{1} k = 1$ *and* $\frac{1}{2} \cdot 1 \cdot (1+1) = \frac{1}{2} \cdot 1 \cdot (2) = 1$.
2.***Heredity :*** *Suppose that for a given* $n \in \mathbb{N}^*$ *we have* $A(n) = B(n)$. *Let's prove that* $A(n+1) =$

$B(n+1)$ *i.e.* $\sum_{k=1}^{n+1} k = \frac{1}{2}(n+1)(n+2)$.

$$\begin{aligned}
A(n+1) &= \sum_{k=1}^{n+1} k \\
&= 1+2+3+4+\cdots+n+(n+1) \\
&= \sum_{k=1}^{n} k + (n+1) \\
&= A(n) + (n+1) \\
&= B(n) + (n+1) \text{ because of the induction hypothesis} \\
&= \frac{1}{2}n(n+1) + (n+1) \\
&= \frac{n(n+1)+2(n+1)}{2} \\
&= \frac{n^2+3n+2}{2} \\
&= \frac{(n+1)(n+2)}{2} \\
&= \frac{1}{2}(n+1)(n+2) \\
&= B(n+1)
\end{aligned}$$

3.***Conclusion :*** $\forall n \in \mathbb{N}^*, \sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$

8. *We will show by induction that :* $\forall n \in \mathbb{N}$, $7^n - 1$ *is divisible by* 6.
   1.***Initialization :*** *For* $n = 0$ *we have* $7^0 - 1 = 1 - 1 = 0$ *and* 0 *is divisible by* 6.
   2.***Heredity :*** *Suppose that for a given* $n \in \mathbb{N}$, $7^n - 1$ *is divisible by* 6. *Let's show that* $7^{n+1} - 1$ *est divisible by* 6.

$$\begin{aligned}
7^{n+1} - 1 &= 7 \cdot 7^n - 1 \\
&= (6+1) \cdot 7^n - 1 \\
&= 6 \cdot 7^n + 7^n - 1
\end{aligned}$$

   $6 \cdot 7^n$ *is divisible by* 6 *and by induction hypothesis* $7^n - 1$ *is also divisible by* 6 *then* $6 \cdot 7^n + 7^n - 1$ *is divisible by* 6, *that is* $7^{n+1} - 1$ *is divisible by* 6.
   3.***Conclusion :*** $\forall n \in \mathbb{N}$, $7^n - 1$ *is divisible by* 6.

9. *Now we show that :* $\forall n \in \mathbb{N}$, $4^n + 6n - 1$ *is a multiple of* 9 *(supp).*
   Put $P(n)$: $4^n + 6n - 1$ *is a multiple of* 9.
   1.***Initialization :*** *For* $n = 0$ *we have* $P(0) : 4^0 + 6 \cdot 0 - 1 = 0$ *and* 0 *is a multiple of* 9. $P(0)$ *is then verified.*
   2.***Heredity :*** *Suppose that* $P(n)$ *is verified for a given* $n \in \mathbb{N}$, *that is* $\exists k_n \in \mathbb{N}$, $4^n + 6n - 1 = 9k_n$. *Let's show that* $P(n+1)$ *is also verified.* $P(n+1)$ *is given by :* $4^{n+1} + 6(n+1) - 1$ *is a multiple*

of 9.

$$\begin{aligned}
4^{n+1}+6(n+1)-1 &= 4\cdot 4^n+6n+6-1\\
&= (1+3)\cdot 4^n+6n-1+6\\
&= {\color{red}4^n+6n-1}+3\cdot 4^n+6\\
&= 9\cdot k+3\cdot 4^n+6 \textit{ because of the induction hypothesis}\\
&= 9\cdot k_n+3({\color{green}9k_n-6n+1})+6\\
&\quad \textit{the formula } {\color{green}9k_n-6n+1} \textit{ is derived from the induction hypothesis}\\
&= 9(k_n+3k_n-2n+1)\\
&= 9(4k_n-2n+1)\\
&= 9k_n^{'} \textit{ with } k_n^{'}=4k_n-2n+1\in\mathbb{N}
\end{aligned}$$

Hence $P(n+1)$ *is verified.*
3.***Conclusion :*** $\forall n\in\mathbb{N}$, 4^n+6n-1 *is a multiple of* 9