Prerequisites

The student must know the scalar product and the vector product, as well as the total derivative and the partial derivative.

Objective

To provide some of the mathematical definitions needed to perform calculations in order to begin the Electricity program.

Summary

Part I: Elementary movements in different coordinate systems

1. **Cartesian coordinates : (x, y, z)**

$$\overrightarrow{dr} = \overrightarrow{dOM} = dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k} \qquad (1)$$

2. **Polar coordinates : (r, Θ)**

$$\overrightarrow{dr} = dr\overrightarrow{u_r} + rd\theta\overrightarrow{u_\theta} \qquad (2)$$

3. **Cylindrical coordinates : (ρ, Θ, z)**

$$\overrightarrow{dr} = d\rho\overrightarrow{u_\rho} + \rho d\theta\overrightarrow{u_\theta} + dz\overrightarrow{u_z} \qquad (3)$$

4. **Spherical coordinates : (r, Θ, φ)**

$$\overrightarrow{dr} = \overrightarrow{dOM} = dr\overrightarrow{U_r} + rsin\varphi d\theta\overrightarrow{U_\theta} + rd\varphi\overrightarrow{U_\varphi} \qquad (4)$$

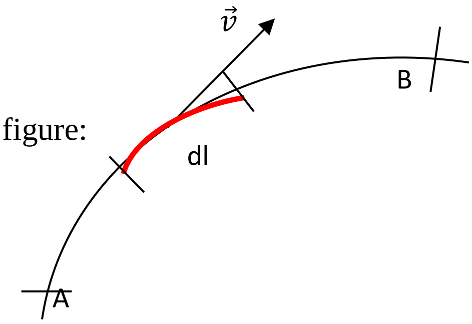
Part II: Differential operators

1. Notion of Field

Faynman defined a field as any physical quantity that takes on a different value at any point in space. It depends on the coordinates of the point under consideration. A distinction is made between scalar and vector fields. Temperature is a scalar field, noted T(x,y,z), and can also vary with time. Another example is the velocity field of a flowing liquid, noted v(x,y,z), which also varies with time; this is a vector field.

A vector field $V(xyz)$ is the datum of three scalar fields, and can be written in different bases, namely Cartesian $(\vec{\imath}, \vec{\jmath}, \vec{k})$, polar, $(\overrightarrow{Ur}, \overrightarrow{U\theta})$, cylindrical $(\overrightarrow{U\rho}, \overrightarrow{U\theta}, \vec{k})$, or spherical $(\overrightarrow{Ur}, \overrightarrow{U\theta}, \overrightarrow{U\varphi})$.

2. Circulation of a vector جولان شعاع



Let be a vector $\vec{v}$ and a parcel AB given by the following figure:

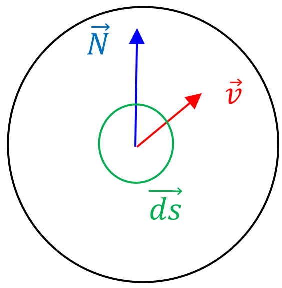
The elementary circulation of the vector $\vec{v}$ along of a displacement element dl is given by the relation:

$$d\varsigma = \vec{v}.\overrightarrow{dl} \qquad (5)$$

So the circulation of the vector $\vec{v}$ along the path AB is:

$$\varsigma = \int_{AB} \vec{v}.\overrightarrow{dl} \qquad (6)$$

3. Flux of a vector field تدفق حقل شعاعي



Let be a vector $\vec{v}$ and a surface (s) given by the following figure:

The elementary flux of the vector $\vec{v}$ across the elementary surface ds is given by the relation:

$$d\phi = \vec{v}.\overrightarrow{ds} \qquad (7)$$

The vector $\overrightarrow{ds}$ written by :

$$\overrightarrow{ds} = ds.\vec{N} \qquad (8)$$

With $\vec{N}$ is the vector normal to the surface (s) so: $d\phi = \vec{v}.\vec{N}ds$

So the total flux of $\vec{v}$ across the surface (s) will be:

$$\phi = \iint_{(s)} \vec{v}.\overrightarrow{ds} = \iint_{(s)} \vec{v}.\vec{N}ds \qquad (9)$$

4. Differential operators

The Nabla operator defined in Cartesian coordinates as:

$$\vec{\nabla}= \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k} \qquad (10)$$

It has the dual property of behaving both as a derivation operator and as a vector in vector product or mixed product formulas. This makes it possible to express many gradient, divergence and rotational relationships directly, without having to go through the components. This operator can also be expressed in the cylindrical and spherical systems.

4.1. Gradient operator (تدرج)

It is a vector whose components are the partial derivatives of a scalar field f(x,y,z) with respect to the coordinates under consideration, its symbol is :

$$\overrightarrow{\text{grad}}\,\text{f} = \vec{\nabla}.f = \frac{\partial f}{\partial x}\vec{\imath} + \frac{\partial f}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k} \qquad (11)$$

<u>Notes</u>:

- The gradient represents the variability of this function in the vicinity of a point. The gradient is the rate of growth in this direction.
- The gradient is a vector applied to a scalar field.
- The gradient of a function f(x,y,z) is a vector.

If f(x,y,z) then :

$$\overrightarrow{grad}\, f(x,y,z) = \vec{\nabla}\, f(x,y,z) = \frac{\partial}{\partial x} f(x,y,z)\,\vec{\imath} + \frac{\partial}{\partial y} f(x,y,z)\,\vec{\jmath} + \frac{\partial}{\partial z} f(x,y,z)\,\vec{k} \qquad (12)$$

- With, $\frac{\partial}{\partial x} f(x,y,z)$ the partial derivative of f(x,y,z) with respect to x, considering y and z as constants.
- $\frac{\partial}{\partial y} f(x,y,z)$ the partial derivative of f(x,y,z) with respect to y, considering x and z as constants.

- $\frac{\partial}{\partial z} f(x, y, z)$ the partial derivative of f(x,y,z) with respect to z, considering y and x as constants.

Example :

$$U = x^2 + xy + xyz$$

$$\overrightarrow{grad}\, U = \vec{\nabla}\, U = \frac{\partial}{\partial x} U\, \vec{\imath} + \frac{\partial}{\partial y} U\, \vec{\jmath} + \frac{\partial}{\partial z} U\, \vec{k} = (2x + y + yz)\, \vec{\imath} + (x + xz)\vec{\jmath} + xy\, \vec{k}$$

4.2. Divergence operator

The divergence of a vector field is a scalar defined by applying the Nabla operator to a vector field $\vec{v}\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$, calculated from the following expression :

$$div\, \vec{v} = \vec{\nabla}.\vec{v} = \left(\frac{\partial v_x}{\partial x}\right)_{y,z=cst} + \left(\frac{\partial v_y}{\partial y}\right)_{x,z=cst} + \left(\frac{\partial v_z}{\partial z}\right)_{x,y=cst} = \text{Scalar} \quad (13)$$

Example :

$$\vec{v} = 2x\, \vec{\imath} + xy^2\vec{\jmath} + z^2\, \vec{k}$$

$$\left(\frac{\partial(xy^2)}{\partial y}\right)_{x,z=cst} = x.2y$$

$$div\, \vec{v} = \vec{\nabla}.\vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 2 + 2xy + 2z$$

4.3. Rotary Operator

It is a vector that applies to a vector (vector field) $\vec{v}\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$; calculated from the following expression:

$$\overrightarrow{Rot}\, \vec{v} = \vec{\nabla} \wedge \vec{v} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \vec{\imath} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_y & v_z \end{vmatrix} - \vec{\jmath} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ v_x & v_z \end{vmatrix} + \vec{k} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ v_x & v_y \end{vmatrix} \quad (14)$$

$$\vec{\nabla} \wedge \vec{B} = \vec{\imath}\left(\frac{\partial}{\partial y} v_z - \frac{\partial}{\partial z} v_y\right) - \vec{\jmath}\left(\frac{\partial}{\partial x} v_z - \frac{\partial}{\partial z} v_x\right) + \vec{k}\left(\frac{\partial}{\partial x} v_y - \frac{\partial}{\partial y} v_x\right) \quad (15)$$

Note :

If $\overrightarrow{Rot}\, \vec{v} = \vec{0}$ the field is non-rotating.

If $\overrightarrow{Rot}\, \vec{v} \neq \vec{0}$ the field is rotating.

4.4. The Laplacian

The Laplacian is defined as the divergence of the gradient, denoted Δ. A distinction is made between the scalar Laplacian if it is used for a scalar field; it gives a scalar, and the vector Laplacian if it is applied to a vector field; it gives a vector.

$$\Delta = \vec{\nabla}.\vec{\nabla} = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad (16)$$

By associating the Laplacian with a function, we get:

$$\Delta\emptyset = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = Scalar \qquad (17)$$

By associating the Laplacian with a vector, we get:

$$\Delta\emptyset = \frac{\partial^2 \vec{v}}{\partial x^2} + \frac{\partial^2 \vec{v}}{\partial y^2} + \frac{\partial^2 \vec{v}}{\partial z^2} = Vector \qquad (18)$$

Note :

The Laplacian expresses the dynamic variation in space of a field or function.

5. Creen's formula

In a closed surface, we have: $$\iiint div\, \vec{v} dv = \oiint \vec{v}.\overrightarrow{ds} \qquad (19)$$

6. Stocks Formula

The circulation of a vector along a closed contour is equal to the flow of its rotational vector through the surface resting on that contour.

$$\int \vec{v}.\overrightarrow{dl} = \iint \overrightarrow{Rot}\, \vec{v}.\overrightarrow{ds} \qquad (20)$$

<u>Application :</u>

Let $\vec{A} = 2xyz\vec{\imath} + (2x^2 - y)\vec{\jmath} - yz^2\vec{k}$ et $\phi = x^2y + 2y^2z^3$

Give on the point (1,0,0): $\overrightarrow{grad}\,\phi$, $div\vec{A}$, $\overrightarrow{Rot}\vec{A}$.

Solution :

- The gradient :

$$\overrightarrow{grad}\,\phi(x,y,z) = \vec{\nabla}\,\phi(x,y,z)$$

$$= \frac{\partial}{\partial x}\phi(x,y,z)\,\vec{\imath} + \frac{\partial}{\partial y}\phi(x,y,z)\,\vec{\jmath} + \frac{\partial}{\partial z}\phi(x,y,z)\,\vec{k}$$

$$\phi = x^2y + 2y^2z^3$$

$$\frac{\partial}{\partial x}\phi(x,y,z) = 2xy$$

$$\frac{\partial}{\partial y}\phi(x,y,z) = x^2 + 4yz^3$$

$$\frac{\partial}{\partial z}\phi(x,y,z) = 6y^2z^2$$

$\overrightarrow{grad}\,\phi(x,y,z) = 2xy\,\vec{\imath} + (x^2 + 4yz^3)\,\vec{\jmath} + 6y^2z^2\,\vec{k}$

To the point (1,0,0) we have: $\overrightarrow{grad}\,\phi(x,y,z) = \vec{\jmath}$

- $div\,\vec{A} = \vec{\nabla}.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

$\vec{A} = 2xyz\vec{\imath} + (2x^2 - y)\vec{\jmath} - yz^2\vec{k}$

$A_x = 2xyz, \quad A_y = (2x^2 - y), \quad A_z = -yz^2$

$\frac{\partial A_x}{\partial x} = 2yz, \qquad \frac{\partial A_y}{\partial y} = -1, \qquad \frac{\partial A_z}{\partial z} = -2yz$

$div\,\vec{A} = 2yz - 1 - 2yz = -1$

- $\overrightarrow{Rot}\,\vec{A} = \vec{\nabla} \wedge \vec{A} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz & (2x^2 - y) & -yz^2 \end{vmatrix}$

$$\vec{\imath}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2x^2 - y) & -yz^2 \end{vmatrix} - \vec{\jmath}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ 2xyz & -yz^2 \end{vmatrix} + \vec{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 2xyz & (2x^2 - y) \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(-yz^2) - \frac{\partial}{\partial z}(2x^2 - y)\right)\vec{\imath} - \left(\frac{\partial}{\partial x}(-yz^2) - \frac{\partial}{\partial z}(2xyz)\right)\vec{\jmath}$$

$+\left(\frac{\partial}{\partial x}(2x^2 - y) - \frac{\partial}{\partial y}(2xyz)\right)\vec{k}$

$\overrightarrow{Rot}\,\vec{A} = (-z^2 - 0)\vec{\imath} - (0 - 2xy)\vec{\jmath} + (4x - 2xz)\vec{k}$

$\overrightarrow{Rot}\,\vec{A} = (-z^2)\vec{\imath} + (2xy)\vec{\jmath} + (4x - 2xz)\vec{k}$

To the point (1,0,0) we have: $\overrightarrow{Rot}\,\vec{A} = 4\,\vec{k}$