Number systems

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Outline

Definition

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The number system is a mathematical system used to represent and count numbers.

Definition

Let b be an integer; $b \geq 2$. Any natural number N can be expressed as a sum of terms of the form $a_i \cdot b^i$, where the numbers i are natural integers, and the numbers a_i are natural integers between 0 and $b - 1$. The integers a_i are the digits (or symbols) and the integer b is called radix or base.
We can express N in the base b as
$(N)_b = (a_n a_{n-1} \cdots a_1 a_0)_b$ and the decomposition $(N)_b = a_n \cdot b^n + a_{n-1} \cdot b^{n-1} + \cdots + a_1 \cdot b^1 + a_0 \cdot b^0$, will be called the polynomial form of the number N.

Examples

Example

We consider the decimal number $2345 = 2.10^3 + 3.10^2 + 4.10^1 + 5.10^0$, we say that 2345 is written in radix-10 or in base 10

Example

We have $17 = 16 + 1 = 1 \cdot 2^4 + 1 \cdot 2^0 = 1 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$, we say that 10001_2 is a a representation of 17 in radix-2 or in base 2.

The decimal number system is a radix-10 number system and has 10 different digits or symbols. These are $0, 1, 2, 3, 4, 5, 6, 7, 8$ and 9.

Example

6935 can be expressed as:
$6935 = 6 \times 10^3 + 9 \times 10^2 + 3 \times 10^1 + 5 \times 10^0$

Binary Number System

Because digital circuits work with only two voltage states, it is logical to use the binary number system to keep track of information. The binary number system with $'0'$ and $'1'$ as the two independent digits. The procedure for wrinting higher order binary numbers after $'1'$ is similar to the one explained in the case of the decimal number system.

Example

The first 16 natural numbers in the binary number system.

Example

A binary number such as 11011_2 (27_{10}) can be expressed as successive powers of 2.

$$11011_2 = 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0.$$

Bit

Two other number systems used in digital electronics include the octal and hexadecimal systems.

The octal number system has a radix of 8 and therefore has eight distinct digits; $0, 1, 2, 3, 4, 5, 6$ and 7.

The hexadecimal number system is a radix-16 number system and its 16 basic digits are $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E$ and F; where $A = 10, B = 11, C = 12, D = 13, E = 14$ and $F = 15$.

Example

An octal number such as 2107_8 can be expressed as successive powers of 8.

$$2107_8 = 2 \cdot 8^3 + 1 \cdot 8^2 + 0 \cdot 8^1 + 7 \cdot 8^0 = 1095.$$

Bit and Byte

Definition

Bit is an abbreviation of the term *'binary digit'*; it is the smallest unit of information. It is either $'0'$ or $'1'$.

Definition

Byte is a string of eight bits. The byte is the basic unit of data operated upon as a single unit in computers.

Definition

A computer word is again a string of bits whose size, called the *'word lengh'* or *'word size'* is fixed for a specified computer. The word lenght may equal one byte, two bytes, four bytes or be even larger.

MSB and LSB

Example

We can represent 128 in binary with 1 byte (8 bits),

$$128 = 10000000_2.$$

Definition

In a binary number, the bit furthest to the left is called the most significant bit (MSB) and the bit furthest to the right is called the least significant bit (LSB).

Example

$$\underbrace{1}_{\text{MSB}}\,100101\,\underbrace{0}_{\text{LSB}}.$$

To decimal

For an integer N represented by n digits with radix b, the formula for conversion to decimal representation is as follow:

$$(a_{n-1}a_{n-2}\cdots a_2a_1a_0)_b = \sum_{i=0}^{n-1} a_i b^i = N.$$

Example

Convert the binary number 100111_2, the octal number 651_8 and the hexadecimal number $4AC_{16}$ to decimal.

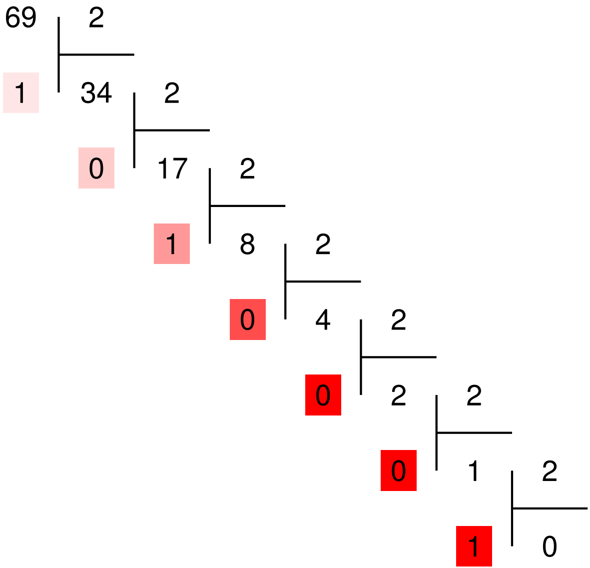
- $100111_2 = 1\cdot 2^5 + 0\cdot 2^4 + 0\cdot 2^3 + 1\cdot 2^2 + 1\cdot 2^1 + 1\cdot 2^0 = 39,$
- $651_8 = 6\cdot 8^2 + 5\cdot 8^1 + 1\cdot 8^0 = 425,$
- $4AC_{16} = 4\cdot 16^2 + 10\cdot 16^1 + 12\cdot 16^0 = 1196.$

From decimal

1. The binary equivalent of decimal number can be found by successively dividing the number by 2 and recording the remainders until the quotient becomes ′0′. The remainders written in reverse order constitute the binary equivalent.
2. The process of decimal to octal conversion is similar to that of decimal to binary conversion. The division here is by 8.
3. The process of decimal to hexadecimal conversion is also similar. Since the haxadecimal number system has a base of 16, the progressive division in this case is 16.

The conversion of any decimal number to a number in base b can be hold by successively dividing the bumber by b and recording the remainders until the quotient becomes ′0′.

Examples



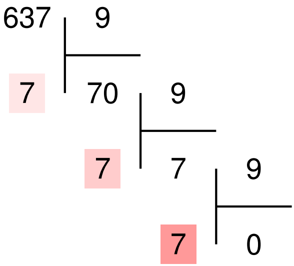
$69 = 1000101_2$

Decimal to Binary-Octal-Hexadecimal

265	8				
1	33	8			
	1	4	8		
		4	0		

$$256 = 411_8$$

Decimal to Binary-Octal-Hexadecimal



$$637 = 777$$

Decimal to Binary-Octal-Hexadecimal

2359	16		
7	147	16	
	3	9	16
		9	0

$$2359 = 937_{16}$$

From octal to binary

An octal number can be converted into its binary equivalent by replacing each octal digit with its three-bit binary equivalent. We take the three-bit equivalent because the base of the octal number system is 8 and it is the third power of the base of the binary number system, i.e 2^3. A binary number can be converted into an equivalent octal number by splitting the number into groups of three bits, starting from the binary point on both sides. The 0s can be added to complete the outside groups if needed.

Example

Let us find the binary equivalent of $(3764)_8$.

$$\underbrace{3}_{011}\underbrace{7}_{111}\underbrace{6}_{110}\underbrace{4}_{100},$$

so $(3764)_8 = (11111110100)_2$.

From binary to octal

Example

Let us find the octal equivalent of $(10011000111)_2$.

$$\underbrace{010}_{2}\ \underbrace{011}_{3}\ \underbrace{000}_{0}\ \underbrace{111}_{7},$$

so $(10011000111)_2 = (2307)_8$.

From Hex to binary

A hexadecimal number can be converted into its binary equivalent by replacing each hex digit with its four-bit binary equivalent. We take the four-bit equivalent because the base of the hexadecimal number system is 16 and it is the fourth power of the base of the binary number system. A given binary number can be converted into an equivalent hexadecimal number by splitting digits into groups of four bits, starting from the binary point on both sides. The 0s can be added to complete the outside groups if needed.

Example

Let us find the binary equivalent of $(3569)_{16}$.

$$\underbrace{3}_{0011}\ \underbrace{5}_{0101}\ \underbrace{6}_{0110}\ \underbrace{9}_{1001},$$

so $(3569)_{16} = (0011010101101001)_2$.

From binary to Hex

Example

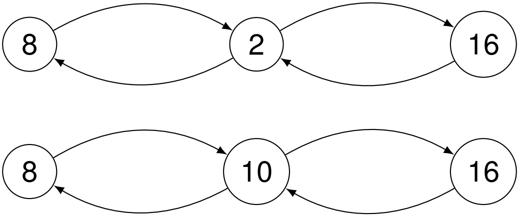
Let us find the hex equivalent of $(1011010001100101 11)_2$.

$$\underbrace{0010}_{2}\ \underbrace{1101}_{D}\ \underbrace{0001}_{1}\ \underbrace{1001}_{9}\ \underbrace{0111}_{7},$$

so $(101101000110010111)_2 = 2D197_{16}$

For hexadecimal-octal conversion, the given hex number is firstly converted into its binary equivalent which is further converted into its octal equivalent. An alternative approach is firstly to convert the given hexadecimal number into its decimal equivalent and then convert the decimal number into an equivalent octal number. For octal-hexadecimal conversion, the octal number may first be converted into an equivalent binary number and then the binary number transformed into its hex equivalent. The other option is firstly to convert the given octal number into its decimal equivalent and then convert the decimal number into its hex equivalent.

Hex-Octal and Octal-Hex conversions



Examples

Example

Let us find the octal equivalent of $(54F)_{16}$.

$$\underbrace{5}_{0101}\underbrace{4}_{0100}\underbrace{F}_{1111},$$

then $(54F)_{16} = \underbrace{010}_{2}\underbrace{101}_{5}\underbrace{001}_{1}\underbrace{111}_{7} = (2517)_8$

Example

Let us find the hex equivalent of $(472)_8$.

$$\underbrace{4}_{100}\underbrace{7}_{111}\underbrace{2}_{010},$$

then $(472)_8 = \underbrace{0001}\underbrace{0011}\underbrace{1010} = (13A)_{16}$.

Definition

Definition

Fractional number is a number in the form $\frac{p}{q}$,
where p and q are natural numbers and q is not equal to zero.

Example

$\frac{9}{6} = 1.333\cdots$, $\frac{10}{4} = 2.5$ are fractional numbers.

Definition

In radix-b representation, a fractional number N has the form.

$$(N)_b = (a_n a_{n-1} \cdots a_1 a_0 . a_{-1} a_{-2} \cdots a_{-p})_b.$$

Example

$(101111.110111)_2$ is a fractional number in radix-2.

Radix-b to decimal

To convert a fractional number $(N)_b = (a_n a_{n-1} \cdots a_0 a_{-1} a_{-2} \cdots a_{-p})_b$; writing in radix-$b$ representation to decimal, we use the polynomial formula: $(N)_b = a_n \cdot b^n + a_{n-1} \cdot b^{n-1} + \cdots + a_1 \cdot b^1 + a_0 \cdot b^0 + a_{-1} \cdot b^{-1} + a_{-2} \cdot b^{-2} + \cdots + a_{-p} \cdot b^{-p}$.

Example

Let us convert $(1001.101)_2$ to decimal.

$$(1001.101)_2 = 1{\cdot}2^3 + 0{\cdot}2^2 + 0{\cdot}2^1 + 1{\cdot}2^0 + 1{\cdot}2^{-1} + 0{\cdot}2^{-2} + 1{\cdot}2^{-3} = 9.625$$

decimal to radix-b

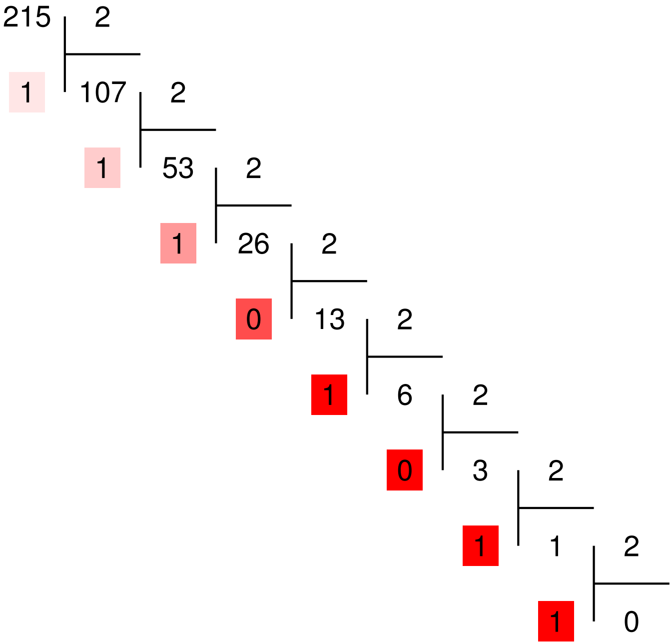
To convert a fractional number from decimal to radix-*b* representation, we must

1. convert the integer part by successively dividing by *b*,
2. convert the fractional part by successively multiply by *b* and retain the digit that becomes an integer.

Example

Convert the fractional number 215.625 to binary representation.

Fractional number conversion



after we first multiply 0.625 by 2

$$
\begin{array}{rcccc}
 & 0 & . & 6 & 2 & 5 \\
\times & & & & & \\
 & & & & & 2 \\
\hline
= & & 1 & . & 2 & 5
\end{array}
$$

The integer part (1) is moved to the right after the decimal point, next we multiply 0.25 by 2;

$$\begin{array}{r} 0\ .\ 2\ 5 \\ \times \qquad\quad 2 \\ \hline = \textcolor{red}{0}\ .\ 5\ 0 \end{array}$$

here, the integer part is 0, then we move it to the right after the last obtained integer part. Then we multiply 0.5 by 2;

$$\begin{array}{cccc} & 0 & . & 5 \\ \times & & & \\ & & & 2 \\ \hline = & \color{red}{1} & . & 0 \end{array}$$

we move 1 after 0 and we obtain

$$215.625 = (11010111.101)_2.$$

To convert a binary number to octal (hexadecimal); we split the digits into groups of three bits (four bits), starting from the decimal point, moving left for the integer part and moving right for the fractional part. The zeros can be added to cmplete the outside groups if needed.

Example

1. Convert 100001101.11001_2 to octal representation.
   $(100001101.11001)_2 = (100\ 001\ 101.110\ 010)_2 = (415.62)_8$
2. Convert $(100011.101)_2$ to hexadecimal representation.
   $(100011.101)_2 = (0010\ 0011.1010)_2 = (23.A)_{16}$

Binary arithmetic-Addition

Basic arithmetic operations include addition, subtraction, multiplication and division. We can write the basic rules of binary addition as follows

$$0 + 0 = 0.$$
$$0 + 1 = 1.$$
$$1 + 0 = 1.$$
$$1 + 1 = 0 \text{ with a carry of } '1' \text{ to the next more significant bit.}$$
$$1 + 1 + 1 = 1 \text{ with a carry of } '1' \text{ to the next more significant bit.}$$

Addition

Example

Perform the following Binary addition.

1. $17 + 15 = (10001)_2 + (1111)_2$

$$\begin{array}{ccccccc} & {}^{1}1 & {}^{1}0 & {}^{1}0 & {}^{1}0 & 1 & (17) \\ + & & & & & & \\ & & 1 & 1 & 1 & 1 & (15) \\ \hline = 1 & 0 & 0 & 0 & 0 & 0 & (32) \end{array}$$

2. $11 + 7 = (1011)_2 + (111)_2$

$$\begin{array}{cccccc} & {}^{1}1 & {}^{1}0 & {}^{1}1 & 1 & (11) \\ + & & & & & \\ & & 1 & 1 & 1 & (7) \\ \hline = 1 & 0 & 0 & 1 & 0 & (18) \end{array}$$

Subtraction

The basic principles of binary subtraction include the following

$0 - 0 = 0.$

$1 - 0 = 1.$

$1 - 1 = 0.$

$0 - 1 = 1$ with a borrow of 1 from the next more significant bit.

Subtraction-Example

Example

Perform the following Binary subtraction.

	1	$^{1}1$	$^{1}0$	1	(13)
-					
	$_{1}$	$_{1}1$	1	0	(6)
=	0	1	1	1	(7)

Multiplication

The basic rules of binary multiplication are listed as follows.

$$0 \times 0 = 0.$$
$$0 \times 1 = 0.$$
$$1 \times 0 = 0.$$
$$1 \times 1 = 1.$$

The method for multiplication of larger-bit binary numbers is similar to the multiplication in the case of decimal numbers.

Multiplication-Example

Example

Perform the following Binary multiplication.
$11 \times 5 = (1011)_2 \times (101)_2$

$$
\begin{array}{cccccccc}
 & & & 1 & 0 & 1 & 1 & (11) \\
\times & & & & 1 & 0 & 1 & (5) \\
\hline
 & & & 1 & 0 & 1 & 1 & \\
+ & & {}^{1}0 & 0 & 0 & 0 & . & \\
+ & 1 & 0 & 1 & 1 & . & . & \\
\hline
= & 1 & 1 & 0 & 1 & 1 & 1 & (55)
\end{array}
$$

The algorithm for binary division is some what similar to decimal division.
The binary division rules are as follows.

$$0 \div 1 = 0.$$
$$1 \div 1 = 1.$$

Division-Example

Example

Perform the following Binary division. $13 \div 5 = (1101)_2 \div (101)_2$

$$\begin{array}{r|l} 11^{1}01 & 101 \\ \cline{2-2} -1_{1}01_{\downarrow} & \quad 10 \\ \hline \\ 0011 & \end{array}$$