University of Tlemcen
Faculty of Sciences

Academic year: 2024/2025
Mathematics Department

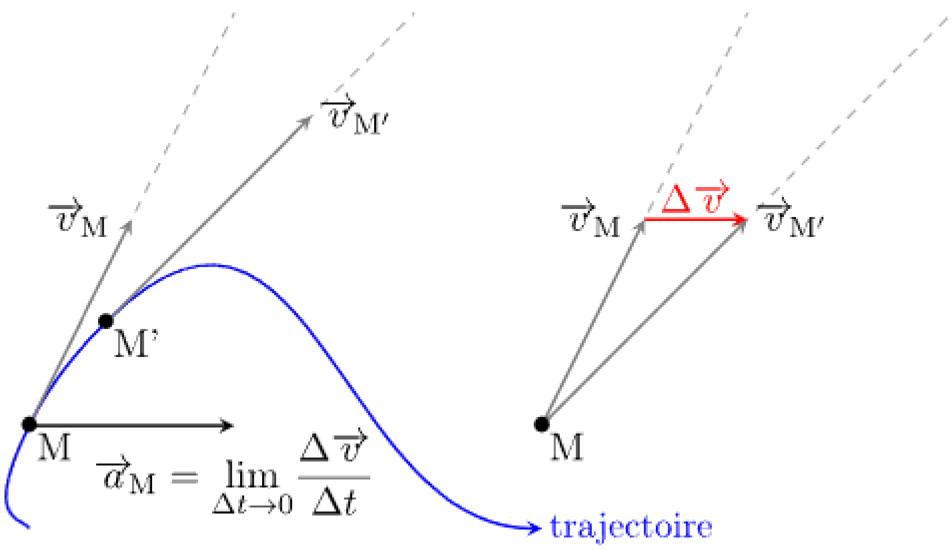
1ST YEAR LMD-M AND MI

COURSE OF MECHANICS

OF THE MATERIAL POINT

Chapter III: Kinematics of material point

Prepared by : Ms Hadjou Belaid Zakia



Summary

1. Introduction

The theory of General Relativity invented by A. Einstein in 1915 is a relativistic theory of gravitation. This theory challenges the idea of an inert Euclidean space, independent of its material content. Kinematics studies the movement of a material point independently of the causes that give rise to it. It is based on a Euclidean description of space and absolute time. The material point is any material body whose dimensions are theoretically zero and practically negligible in relation to the distance it travels. The state of movement or rest of a body are two essentially relative notions: for example, a mountain is at rest in relation to the earth, but in movement in relation to an observer looking at the earth from afar, for whom the globe (with all that it contains) is in perpetual movement. In this course, we illustrate the notions of velocity and acceleration by restricting ourselves to movements in the plane.

2. Reference System مرجع

The concept of motion is relative. A body can be in motion with respect to one object and at rest with respect to another (relative motion), hence the necessity of choosing a reference frame. A reference frame is a system of coordinate axes linked to an observer.
This study of motion is carried out in two forms:

- **Vectorial:** using vectors: position $\overrightarrow{OM}$, velocity $\vec{v}$, and acceleration $\vec{a}$.
- **Algebraic:** by defining the equation of motion along a given trajectory.

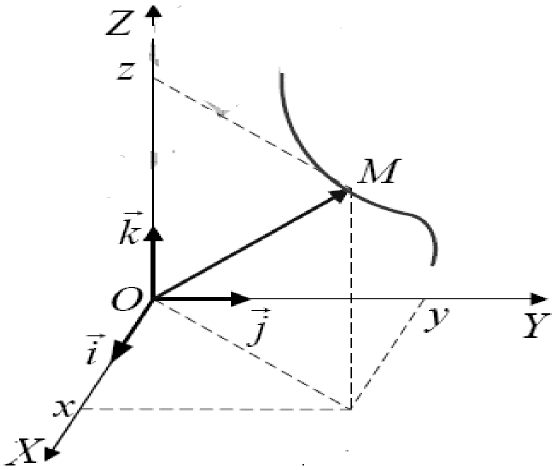
3. Characteristics of a movement

3.1. Vector position and time equationشعاع الموضعي و المعادلة الزمنية للحركة

We define the position of a material point M in a reference frame by the position vector $\overrightarrow{OM}$, where O is a fixed point and serves as the origin of the reference frame. The components of point M or the vector $\overrightarrow{OM}$ are given in the chosen coordinate system's basis (cartesian coordinates, polar coordinates, etc.).
The point M moves through time, and this movement is described by an equation known as the "time equation" (معادلة زمنية), translated as the "time equation."

3.2. Trajectory المسار

The trajectory is the geometric path of successive positions occupied by the material point over time with respect to the considered reference system.



Example:

The position of a material point M identified by its coordinates (x, y, z) at time t in a coordinate system R (O, $\vec{\imath}$, $\vec{\jmath}$, $\vec{k}$) with a position vector:

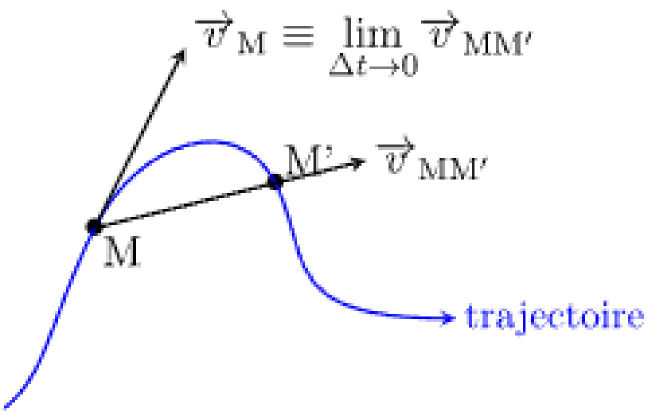
$$\overrightarrow{OM} = (t-1)\vec{\imath} + \frac{t^2}{2}\vec{\jmath}$$

$\overrightarrow{OM} = (t-1)\vec{\imath} + \frac{t^2}{2}\vec{\jmath} \Rightarrow \begin{cases} x = t-1 \\ y = \frac{t^2}{2} \end{cases}$ so $t=x+1$

$\Rightarrow \boldsymbol{y} = \frac{(x+1)^2}{2}$ it's a trajectory equation of the material point.

3.3. Velocity vector شعاع السرعة

Consider a mobile that is located at position M(t) at time t, and it evolves at the point M'(t+Δt) at instant (t+Δt).



- The **average velocity** السرعة المتوسطة between the two instants t and t+Δt is called:

$$\overrightarrow{v_{moy}} = \frac{\overrightarrow{MM'}}{(t+\Delta t)-t} = \frac{\overrightarrow{MM'}}{\Delta t}$$

- If the time interval Δt is very small (Δt→0), we then refer to it as **instantaneous velocity** السرعة اللحضية **:**

$$\vec{v} = \lim_{\Delta t \to 0} \overrightarrow{v_{moy}} = \lim_{\Delta t \to 0} \frac{\overrightarrow{MM'}}{\Delta t}$$

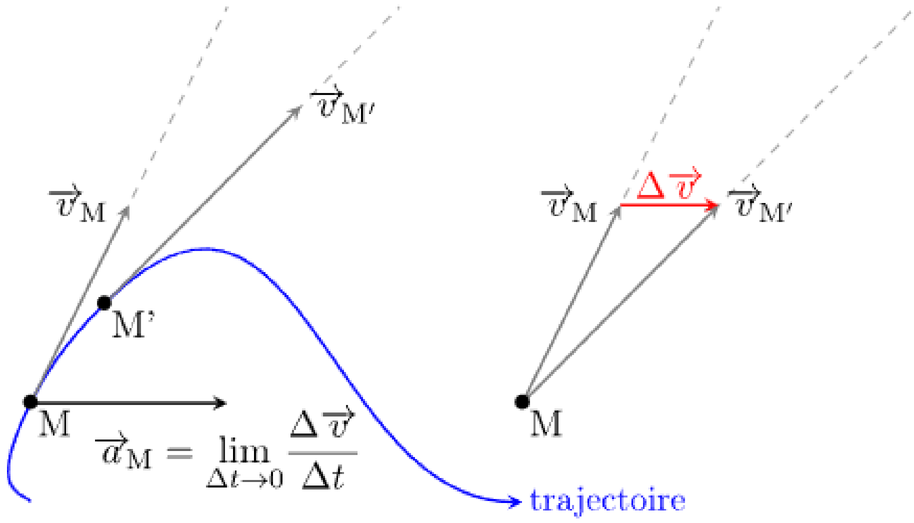
$$\overrightarrow{MM'} = \overrightarrow{MO} + \overrightarrow{OM'} = \overrightarrow{OM'} - \overrightarrow{OM} = \Delta\overrightarrow{OM}$$

So:

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\overrightarrow{OM}}{\Delta t} \Rightarrow \boldsymbol{\vec{v}} = \frac{\boldsymbol{d\overrightarrow{OM}}}{\boldsymbol{dt}}$$

3.4. Acceleration vector شعاع التسارع

When velocity varies over time v=f(t), point M is subjected to an acceleration.



- The average acceleration *التسارع المتوسط* is written:

$$\overrightarrow{a_{moy}} = \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{(t+\Delta t)-t} = \frac{\Delta\vec{v}(t)}{\Delta t}$$

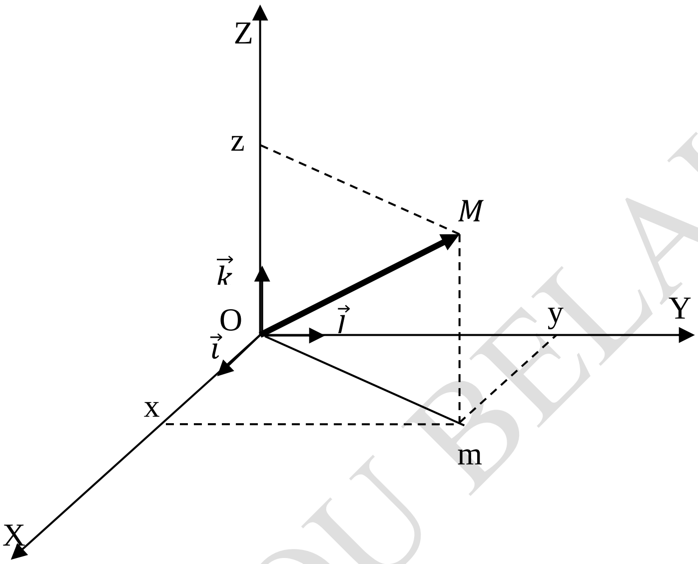
- When the time is very small $\Delta t \to 0$ instantaneous acceleration *التسارع اللحضي* is written by :

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\overrightarrow{OM}}{\Delta t} \Rightarrow \boldsymbol{\vec{a}} = \frac{\boldsymbol{d\vec{v}(t)}}{\boldsymbol{dt}} = \frac{\boldsymbol{d^2\overrightarrow{OM}}}{\boldsymbol{dt^2}}$$

4. Expression of velocity and acceleration in different coordinate systems

4.1. Cartesian coordinate الاحداثيات الكارتيزية

Let's consider point M in space, identified by its coordinates (x, y, z) in the orthonormal coordinate system (Oxyz) with unit vectors $\vec{i},\ \vec{j},\ \vec{k}$.



- **Position vector**

$$\overrightarrow{OM} = x\vec{i} + y\vec{j} + z\vec{k}$$

- **Velocity vector**

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} \quad \Rightarrow \begin{cases} v_x = \dfrac{dx}{dt} \\ v_y = \dfrac{dy}{dt} \\ v_z = \dfrac{dz}{dt} \end{cases}$$

The velocity module is written: $|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$

Note: The magnitude of the velocity, equal to |v|, is called the speed. In S.I. units, v is expressed in (m/s) = (m.s^{-1}).

- **Acceleration vector**: $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\overrightarrow{OM}}{dt^2} \Rightarrow \begin{cases} a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} \\ a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2} \\ a_z = \frac{dv_z}{dt} = \frac{d^2z}{dt^2} \end{cases}$

The acceleration module is written:

$$|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

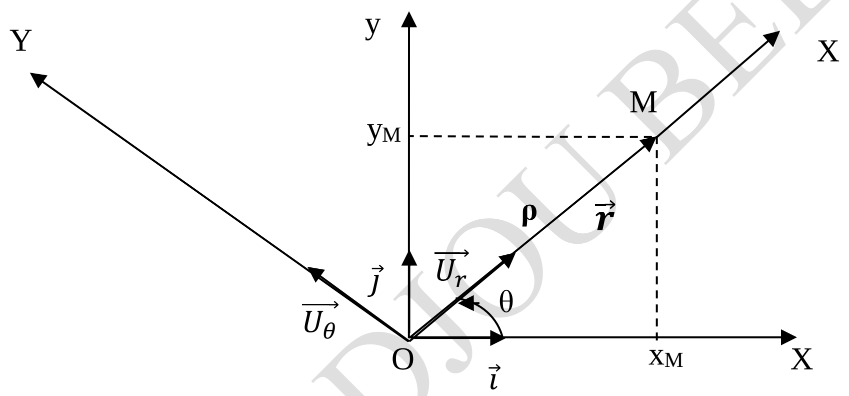
4.2. Polar coordinates الاحداثيات القطبية

When the motion is in a plane, it's also possible to locate the position of point M using its polar coordinates (ρ, θ).

ρ: polar radius ($0 \leq \rho \leq R$)

θ: polar angle ($0 \leq \theta \leq 2\pi$)

Let's consider point M moving in space, identified by its polar coordinates (ρ, θ) in the orthonormal coordinate system (OXY) with unit vectors $\vec{u_r}$, $\vec{u_\theta}$.



- **Position vector**:

$$\overrightarrow{OM} = \rho \vec{U}_r$$

- **Velocity vector :**

$$\vec{\boldsymbol{v}} = \frac{d\overrightarrow{OM}}{dt} = \frac{d\rho}{dt}\vec{U}_r + \rho\,\frac{d\vec{U}_r}{dt}$$

We have : $\frac{d\vec{U}_r}{dt} = \frac{d\vec{U}_r}{dt}\frac{d\theta}{d\theta} = \frac{d\vec{U}_r}{d\theta}\frac{d\theta}{dt}$

With : $\frac{d\vec{U}_r}{d\theta} = \vec{U}_\theta$ donc $\frac{d\vec{U}_r}{dt} = \frac{d\theta}{dt}\vec{U}_\theta$ so $\vec{\boldsymbol{v}} = \frac{d\overrightarrow{OM}}{dt} = \frac{d\rho}{dt}\vec{U}_r + \rho\,\frac{d\theta}{dt}\vec{U}_\theta$

$\Rightarrow \vec{\boldsymbol{v}} = \rho^{\cdot}\vec{U}_r + \rho\,\theta^{\cdot}\vec{U}_\theta$ with $\rho^{\cdot} = \frac{d\rho}{dt}$ and $\theta^{\cdot} = \frac{d\theta}{dt}$

Note: The derivative of a unit vector with respect to an angle is a unit vector perpendicular to the angle in the positive direction.

مشتقة شعاع وحدة بالنسبة إلى الزاوية هي شعاع وحدة عمودي على هذا الاخير في الاتجاه الموجب

- **Acceleration vector**:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\overrightarrow{OM}}{dt^2} = \frac{d^2\rho}{dt^2}\vec{U}_r + \frac{d\rho}{dt}\ \frac{d\vec{U}_r}{dt} + \frac{d\rho}{dt}\frac{d\theta}{dt}\vec{U}_\theta + \rho\ \frac{d^2\theta}{dt^2}\vec{U}_\theta + \rho\ \frac{d\theta}{dt}\frac{d\vec{U}_\theta}{dt}$$

$$\Rightarrow \vec{a} = \frac{d^2\rho}{dt^2}\vec{U}_r + \frac{d\rho}{dt}\ \frac{d\theta}{dt}\vec{U}_\theta + \frac{d\rho}{dt}\frac{d\theta}{dt}\vec{U}_\theta + \rho\ \frac{d^2\theta}{dt^2}\vec{U}_\theta - \rho\ \left(\frac{d\theta}{dt}\right)^2 \vec{U}_r$$

With : $\frac{d\vec{U}_r}{d\theta}\ = \vec{U}_\theta\ et\ \frac{d\vec{U}_\theta}{d\theta}\ = -\vec{U}_r$

$$\text{So : } \vec{a} = \ddot{\rho}\vec{U}_r + 2\dot{\rho}\ \dot{\theta}\vec{U}_\theta + \rho\ \ddot{\theta}\vec{U}_\theta - \rho\ (\dot{\theta})^2\vec{U}_r$$

4.3. Cylindrical Coordinates الاحداثيات الاسطوانية

If the spatial trajectory involves ρ and z playing a specific role in determining the position vector ($\overrightarrow{OM}$); for example, the movement of air molecules in a whirlwind; it is preferable to use cylindrical coordinates (ρ, θ, z).

With:

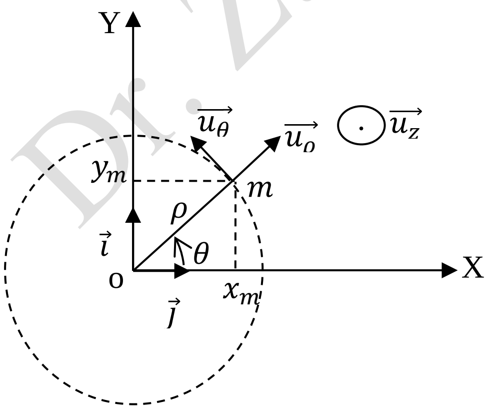
ρ: polar radius

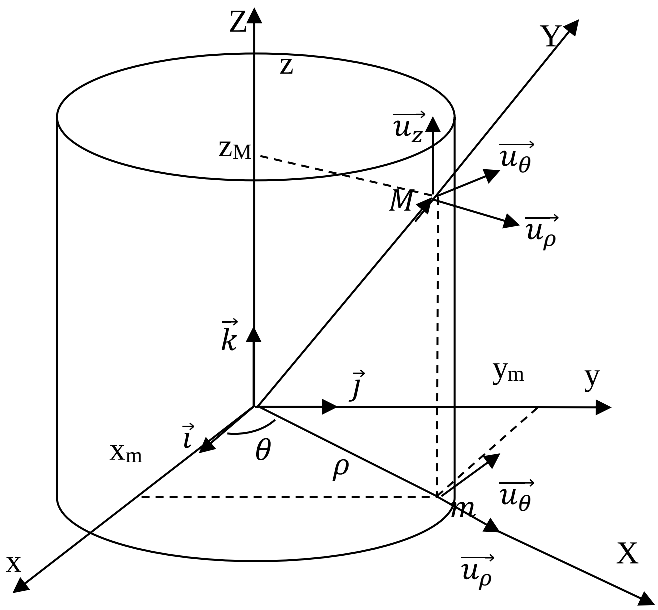
θ: polar angle

z: altitude or height

and $$\begin{cases} \rho = \left|\overrightarrow{Om}\right|,\ \ 0 <\ \rho < R \\ \theta = \left((ox), \overrightarrow{Om}\right), 0 <\ \theta < 2\pi \\ z = z_M\ ,\ \ 0 <\ z < H \end{cases}$$

Where m is the projection of point M onto the plane (Oxy), and R is the radius of the cylinder, and H is the height of the cylinder.





Consider point M moving in space, identified by its cylindrical coordinates (ρ, θ, z) in the orthonormal coordinate system (OXYZ) with unit vectors $\vec{u_\rho}$, $\vec{u_\theta}$, $\vec{u_z}$.

- **Position vector:**

$$\overrightarrow{OM} = \rho\vec{U}_\rho + z\overrightarrow{U_z}$$

- **Velocity vector :**

The velocity in this case is written by: $\vec{\boldsymbol{v}} = \frac{d\overrightarrow{OM}}{dt} = \frac{d\rho}{dt}\vec{U}_r + \rho\,\frac{d\vec{U}_r}{dt} + \frac{dz}{dt}\vec{U}_z + z\,\frac{d\vec{U}_z}{dt}$

$$\frac{d\vec{U}_r}{dt} = \frac{d\vec{U}_r}{dt}\frac{d\theta}{d\theta} = \frac{d\vec{U}_r}{d\theta}\frac{d\theta}{dt}$$

With: $\frac{d\vec{U}_r}{d\theta} = \vec{U}_\theta$ donc $\frac{d\vec{U}_r}{dt} = \frac{d\theta}{dt}\vec{U}_\theta$ et $\frac{d\vec{U}_z}{dt} = \vec{O}$

$$\vec{\boldsymbol{v}} = \frac{d\overrightarrow{OM}}{dt} = \frac{d\rho}{dt}\vec{U}_r + \rho\,\frac{d\theta}{dt}\vec{U}_\theta + \frac{dz}{dt}\vec{U}_z$$

$$\Rightarrow \vec{\boldsymbol{v}} = \dot{\rho}\vec{U}_r + \rho\,\dot{\theta}\vec{U}_\theta + \dot{z}\overrightarrow{U_z}$$

With: $\dot{\rho} = \frac{d\rho}{dt}$, $\dot{\theta} = \frac{d\theta}{dt}$ et $\dot{z} = \frac{dz}{dt}$

- **Acceleration vector:**

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\overrightarrow{OM}}{dt^2} = \frac{d^2\rho}{dt^2}\vec{U}_r + \frac{d\rho}{dt}\,\frac{d\vec{U}_r}{dt} + \frac{d\rho}{dt}\frac{d\theta}{dt}\vec{U}_\theta + \rho\,\frac{d^2\theta}{dt^2}\vec{U}_\theta + \rho\,\frac{d\theta}{dt}\frac{d\vec{U}_\theta}{dt} + \frac{d^2z}{dt^2}\vec{U}_z + \frac{dz}{dt}\frac{d\vec{U}_z}{dt}$$

$$\Rightarrow \vec{a} = \frac{d^2\rho}{dt^2}\vec{U}_r + \frac{d\rho}{dt}\,\frac{d\theta}{dt}\vec{U}_\theta + \frac{d\rho}{dt}\frac{d\theta}{dt}\vec{U}_\theta + \rho\,\frac{d^2\theta}{dt^2}\vec{U}_\theta - \rho\left(\frac{d\theta}{dt}\right)^2\vec{U}_r + \frac{d^2z}{dt^2}\vec{U}_z$$

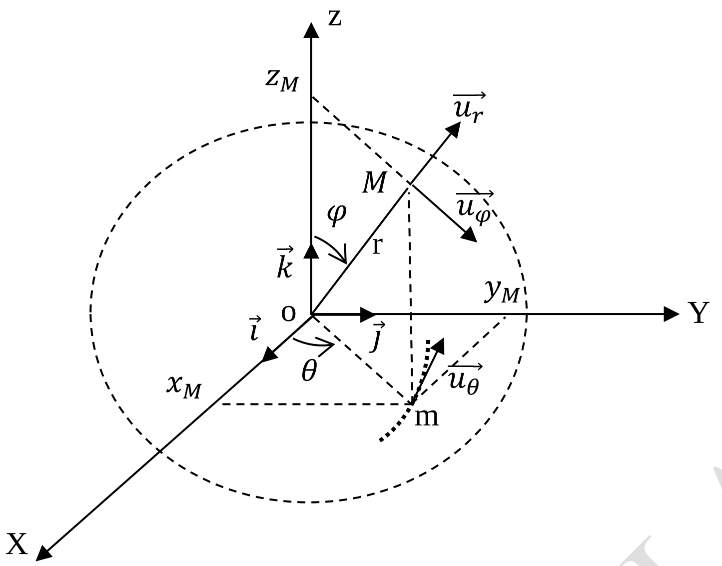
With : $\frac{d\vec{U}_r}{d\theta} = \vec{U}_\theta$, $\frac{d\vec{U}_\theta}{d\theta} = -\vec{U}_r$ et $\frac{d\vec{U}_z}{dt} = \vec{O}$

$$\text{So : } \vec{a} = \ddot{\rho}\vec{U}_r + 2\dot{\rho}\;\dot{\theta}\vec{U}_\theta + \rho\,\ddot{\theta}\vec{U}_\theta - \rho\,(\dot{\theta})^2\vec{U}_r + \ddot{z}\overrightarrow{U_z}$$

4.4. Spherical coordinates الاحداثيات الكروية

When the point O and the distance r separating M and O play a characteristic role, the use of spherical coordinates (r,θ,φ) are best suited in the orthonormed base ($\vec{u_r}$, $\vec{u_\theta}$, $\vec{u_\varphi}$) with:

$$\begin{cases} r = \left|\overrightarrow{OM}\right|, \quad 0 < r < R \\ \theta = \left((ox), \overrightarrow{Om}\right) \quad 0 < \theta < 2\pi \\ \varphi = \left((oz), \overrightarrow{OM}\right) \quad 0 < \varphi < \pi \end{cases}$$



- **Position vector:**

The position vector is written in spherical coordinates (r,θ,φ) by:

$$\vec{r} = \overrightarrow{OM} = r\overrightarrow{U_r}$$

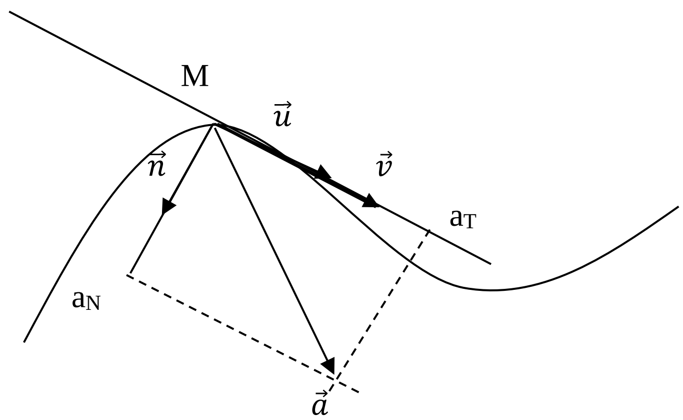
- **Velocity vector :**

The velocity vector is written in spherical coordinates (r,θ,φ) by:

$$\vec{v} = \frac{\overrightarrow{dr}}{dt} = \frac{d\overrightarrow{OM}}{dt} = \dot{r}\overrightarrow{U_r} + r\frac{d\overrightarrow{U_r}}{dt}$$

4.5. Intrinsic coordinates (Frenet frame) احداثيات الحركة المنحنية

We used to work in a fixed frame, but in this case, we study the motion in a moving frame that travels with the moving point "M". This frame is the Frenet frame.



We study the motion in the Frenet frame:

The Frenet frame is a two-dimensional reference frame.

- $\vec{u}$ is the unit vector along the tangent to the trajectory.
- $\vec{n}$ is the unit vector normal to the trajectory and perpendicular to $\vec{u}$, directed towards the center of curvature.
- The position remains unchanged (the frame moves with point M).
- The velocity vector is tangent to the trajectory, and it is written as: $\boldsymbol{\vec{v} = |\vec{v}|\vec{u}}$
- The acceleration vector :

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d|\vec{v}|\vec{u}}{dt} = \frac{d|\vec{v}|}{dt}\vec{u} + |\vec{v}|\frac{d\vec{u}}{dt}$$

$$\frac{d\vec{u}}{dt} = \frac{d\vec{u}}{d\theta}.\frac{d\theta}{dt} = \vec{n}.\omega \quad with \quad \vec{n} = \frac{d\vec{u}}{d\theta} \ and \ \omega = \frac{d\theta}{dt}$$

The acceleration vector is written by: $\vec{a} = a_T\vec{u} + a_N\vec{n}$

So : $\vec{a} = \frac{d|\vec{v}|}{dt}\vec{u} + |\vec{v}|.\vec{n}.\omega$

(the perimeter of a circle (محيط دائرة) $l = 2\pi R$, for the length of a segment (طول قوس) $x = \theta R$; from angular velocity to linear velocity by $\frac{dx}{dt} = R\frac{d\theta}{dt} \Rightarrow v = R\omega$)

Hence:

$\omega = \frac{v}{R}$ with R is the radius of the curvature of the trajectory.

so $\vec{a} = \frac{d|\vec{v}|}{dt}\vec{u} + \frac{v^2}{R}\vec{n}$

The normal acceleration (التسارع الناظمي) and tangential acceleration (التسارع المماس) are written by :

$$\begin{cases} a_T = \frac{d|\vec{v}|}{dt} \\ a_N = \frac{v^2}{R} \end{cases}$$

$$|\vec{a}| = \sqrt{a_x^2 + a_y^2} = \sqrt{a_N^2 + a_T^2}$$

$R \to \infty$ so the trajectory is a line .

R is constant, so the trajectory is circular.

5. Study of some movements

5.1. Rectilinear motion حركة خطية

We have linear motion if the trajectory is a straight line.

We choose a point O as the origin on the trajectory and a unit vector $\vec{\imath}$.

The position of the mobile M, as a function of time, is identified by its abscissa:

$x(t) = \overline{OM(t)}$.

The position vector will be: $\overrightarrow{r(t)} = \overrightarrow{OM(t)} = x(t)\vec{\imath}$

5.1.1. Uniform rectilinear motion حركة مستقيمة منتظمة URM

We have uniform rectilinear motion if the trajectory is a straight line and the velocity vector is constant. This is a motion with zero acceleration $\overrightarrow{a(t)} = \vec{0}$.

The initial conditions to t=0 ; $x=x_0$.

The velocity

$$a = \frac{dv}{dt} = 0 \Rightarrow \int_{v_0}^{v} dv = \int_{0}^{t} 0.dt = [cte]$$

So ; $\boldsymbol{v=v_0=cte}$

The position

$$v = \frac{dx}{dt} = v_0 \Rightarrow \int_{x_0}^{x} dx = \int_{0}^{t} v_0 d\,t = [v_0 t]_0^t = v_0 t$$

So : $\boldsymbol{x=v_0\,t+x_0}$ ***This is the hourly equation of the motion. URM***

5.1.2. Uniformly varied rectilinear motion حركة مستقيمة متغيرة بانتظام UVRM

One has a uniformly varied rectilinear movement if the trajectory is a straight and the acceleration is constant.

The initial conditions to t=0 ; $v=v_0$ and $x=x_0$

The velocity

$$a = \frac{dv}{dt} = a_0 \Rightarrow \int_{v_0}^{v} dv = \int_{0}^{t} a_0 d\,t = [a_0 t]_0^t$$

So $\boldsymbol{v=a_0\,t+v_0}$

The position

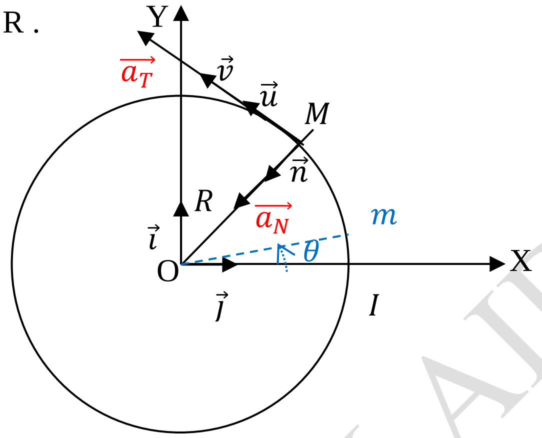
$$v = \frac{dx}{dt} = a_0 t + v_0 \quad \Rightarrow \int_{x_0}^{x} dx = \int_{0}^{t} (a_0 t + v_0) d\,t = \left[\frac{1}{2} a_0 t^2 + v_0 t\right]_0^t$$

so $\boldsymbol{x = \frac{1}{2} a_0 t^2 + v_0 t + x_0}$ ***this is the hourly equation of the motion UVRM***

5.2. Circular motion حركة دائرية

Circular motion is plane motion with constant radius of curvature ρ=R. The trajectory

of the moving object is a circle of radius R .



The position

The moving point travels from point I to point M, thus the trajectory forms an arc $\widehat{IM}$.

By considering an elementary displacement of the moving point from point I to point m, we would have a displacement in the form of an elementary arc Im.

In the right triangle OIm, $\widehat{Im}$=R sinθ

In the right triangle. If θ is so small then $sin\,\theta \approx \theta$.

so **$\widehat{Im}$=Rθ**

The speed

$$v = \frac{d\widehat{Im}}{dt} = R\frac{d\theta}{dt}$$

R is constant, the speed is following the trajectory, so it is written $\vec{v} = v\vec{u}$ so the vector $\vec{u}$ would be following the tangent.

$\boldsymbol{\frac{d\theta}{dt} = \dot{\theta} = \omega}$ ***is the angular velocity*** **السرعة الزاوية**

$$v = R\frac{d\theta}{dt} = R\dot{\boldsymbol{\theta}} = \boldsymbol{R\omega}$$

Note: The relationship between linear velocity and angular velocity is: $\mathbf{v = R\omega}$

The acceleration

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv}{dt}\vec{u} + v\,\frac{d\vec{u}}{dt}$$

$\frac{d\vec{u}}{dt} = \frac{d\vec{u}}{d\theta}\frac{d\theta}{dt}$ *with* $\frac{d\vec{u}}{d\theta} = \overrightarrow{n}$

(with $(\vec{u}, \vec{n})$ the unit vectors in the Fresnet farme and $\frac{d\theta}{dt} = \omega$)

5.2.1. Uniforme circular motion حركة دائرية منتظمة

In this case the angular velocity ω is constant and therefore the linear velocity v is also constant, then $a_T = 0$.

The acceleration in this case is : $\vec{a} = \overrightarrow{a_N} = \frac{v^2}{R}\vec{n}$

5.2.2. Uniformly variable circular motion حركة دائرية متغيرة بانتظام

In this case the angular velocity ω is not constant and therefore the velocity v is not constant also, then $\vec{a} = a_T\vec{u} + a_N\vec{n}$.

The acceleration in this case is: $\vec{a} = \frac{dv}{dt}\vec{u} + \frac{v^2}{R}\vec{n} = R\frac{d\omega}{dt}\vec{u} + R\omega^2\vec{n}$

5.3. Sinusoidal or harmonic motion حركة جيبية

The movement is called sinusoidal or harmonic if its evolution over time is written by the equation:

$$x(t) = A\sin(\omega t + \varphi)$$

A: amplitude, ω: angular frequency, and φ: phase.

$$\omega = \frac{2\pi}{T} = 2\pi f$$

T: period and f: frequency

The speed

$$v(t) = \frac{dx(t)}{dt} = A\omega\cos(\omega t + \varphi)$$

The acceleration

$$a(t) = \frac{dv(t)}{dt} = \frac{d^2x(t)}{dt^2} = -A\,\omega^2\sin(\omega t + \varphi)$$

$$\Rightarrow \frac{d^2x(t)}{dt^2} = -\,\omega^2\,x(t)$$

References :

1. É. Klein. Les tactiques de Chronos, Paris, Flammarion, 2004.
2. T. Damour et al. Relativité, Encyclopædia Universalis,1995.
3. T.Damour. La relativité générale [en ligne, consulté le 2012-03-05]. Disponible sur WWW.CANAL-U.TV.
4. A.F. Bermant, I.G. Aramanovich, Mathematical Analysis (Mir Publishers,1975).
5. D.D. Berkey, Calculus, 2nd Edition (Saunders College, 1988).