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Faculty of Sciences

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Mathematics Department

1ST YEAR LMD-M AND MI

COURSE OF MECHANICS

OF THE MATERIAL POINT

Chaptre IV : Relative motion

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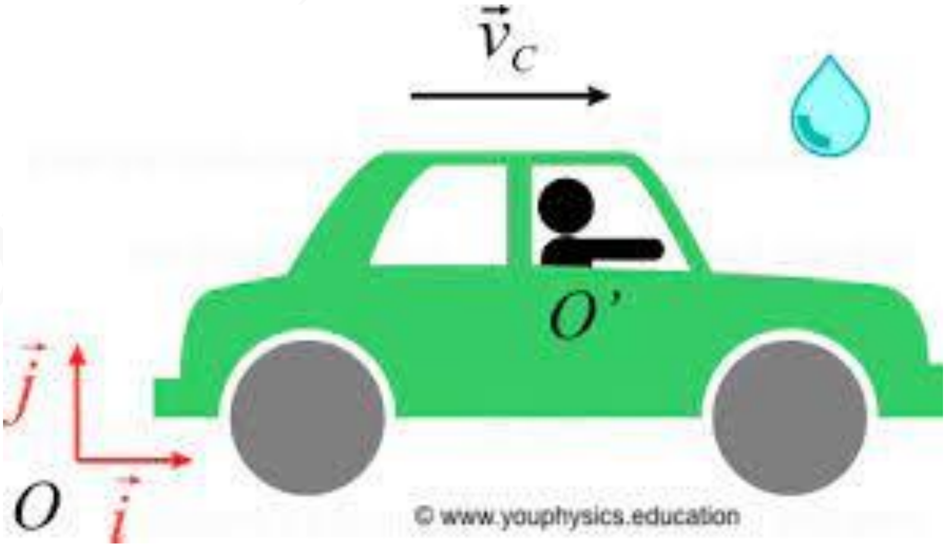


Summary

1. Introduction

State of motion or state of rest are two essentially relative notions, meaning that each of the two states depends on the position of the moving body relative to the body taken as reference frame. All the motions we have studied so far have been in a Galilean frame of reference, i.e. at rest or in uniform rectilinear motion. When two observers linked to two different reference frames are in motion relative to each other, the position, trajectory, velocity and acceleration of the same moving body vary according to the reference frame chosen by the observer.

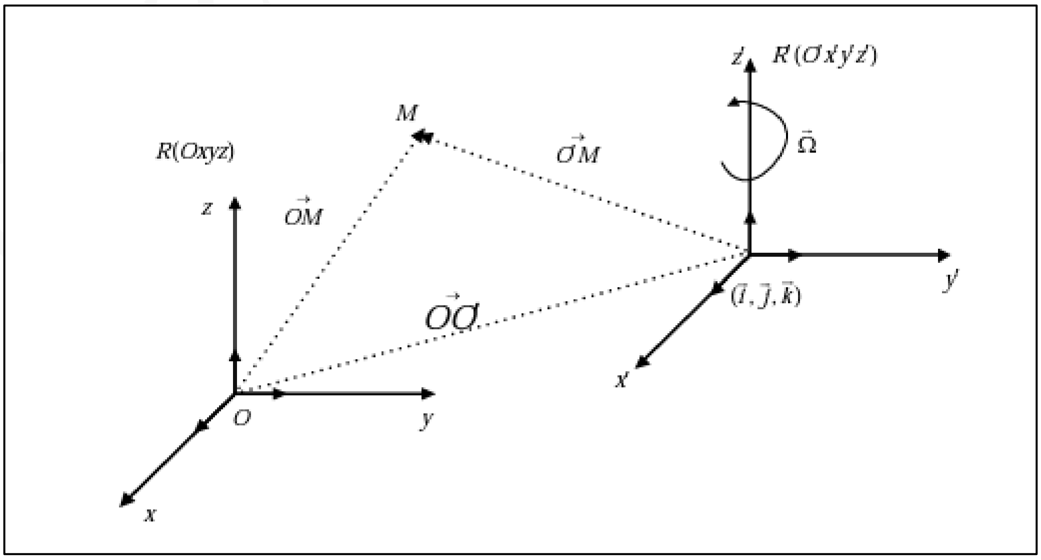
A bus passenger, for example, is in motion relative to an observer seated on the side of the road, whereas he is at rest relative to another observer (a passenger lending the same bus). Clearly, then, the notion of motion or rest is intimately linked to the position of the observer.

To say observer is to say to choose a frame of reference to determine the position, velocity and acceleration of a moving object at each instant.

2. Composition of movements

Let R (O, x, y, z) be a fixed or absolute (Galilean) frame of reference and R' (O', x', y', z') a moving or relative (non-Galilean) frame of reference relative to (R).

It's always useful to know how to determine the position, velocity and acceleration of a material point M in a fixed reference frame if they are known in the other relative reference frame and vice-versa.



Let's associate to the reference frame R (called **absolute reference frame**) the reference frame R $(O,\vec{\imath},\vec{\jmath},\vec{k})$ and to the reference frame R' (called **relative reference frame**) the reference frame R' $(O',\vec{\imath'},\vec{\jmath'},\vec{k'})$.

If M is a movable point in space, defined by coordinates (x, y, z) in the fixed reference frame (R) and by (x', y', z') in the movable reference frame (R').

We will call:

Relative motion: the motion of M relative to (R').

Absolute motion: the motion of M relative to (R).

Entrainment motion: the motion of the moving frame of reference (R') relative to the fixed frame of reference (R).

2.1. Velocity composition

If we know the relative motion, i.e. the motion of the material point considered in the moving frame of reference (relative frame of reference) and that of the moving frame of reference relative to the fixed frame of reference (fixed frame of reference).

We have: $\overrightarrow{OM} = \overrightarrow{OO'} + \overrightarrow{O'M} = \overrightarrow{OO'} + \left(x'\vec{\imath'} + y'\vec{\jmath'} + z'\vec{k'}\right)$

With : $\overrightarrow{OM} = \left(x\vec{\imath} + y\vec{\jmath} + z\vec{k}\right)$ in the fixed reference frame R.

Et $\overrightarrow{O'M} = \left(x'\vec{\imath'} + y'\vec{\jmath'} + z'\vec{k'}\right)$ in the moving reference frame R'.

The velocity is then: $\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} + \frac{dz}{dt}\vec{k}$

$$\frac{d\overrightarrow{OM}}{dt} = \frac{d\overrightarrow{OO'}}{dt} + \frac{d\overrightarrow{O'M}}{dt}$$

$$= \frac{dx'}{dt}\vec{\imath'} + \frac{dy'}{dt}\vec{\jmath'} + \frac{dz'}{dt}\vec{k'} + \frac{d\overrightarrow{OO'}}{dt} + x'\frac{d\vec{\imath'}}{dt} + y'\frac{d\vec{\jmath'}}{dt} + z'\frac{d\vec{k'}}{dt} = \vec{v_a} = \vec{v_r} + \vec{v_e}$$

Avec :

$$\begin{cases} \vec{v_a} = \dfrac{d\overrightarrow{OM}}{dt} = \dfrac{dx}{dt}\vec{\imath} + \dfrac{dy}{dt}\vec{\jmath} + \dfrac{dz}{dt}\vec{k} = \overrightarrow{v(M)}/(R) \\ \vec{v_r} = \dfrac{dx'}{dt}\vec{\imath'} + \dfrac{dy'}{dt}\vec{\jmath'} + \dfrac{dz'}{dt}\vec{k'} = \dfrac{d\overrightarrow{O'M}}{dt}/\,(R') = \overrightarrow{v(M)}/(R') \\ \vec{v_e} = \dfrac{d\overrightarrow{OO'}}{dt} + x'\dfrac{d\vec{\imath'}}{dt} + y'\dfrac{d\vec{\jmath'}}{dt} + z'\dfrac{d\vec{k'}}{dt} = \overrightarrow{v(R')}/(R) \end{cases}$$

$\overrightarrow{\boldsymbol{v_a}}$ represents absolute velocity, the derivative of $\overrightarrow{OM}$ with respect to time in the fixed reference frame.

$\overrightarrow{\boldsymbol{v_r}}$ is the relative speed, i.e. the derivative of $\overrightarrow{O'M}$ with respect to time in the moving reference frame.

$\overrightarrow{\boldsymbol{v_e}}$ is training velocity, this is the derivative of $\overrightarrow{OM}$ with respect to time in the fixed reference frame, considering the moving point M fixed in the moving reference frame (x',y' and z' are constant). It also represents the velocity of the moving frame of reference relative to the fixed frame.

The law of velocity composition is given by:

$$\overrightarrow{\boldsymbol{v_a}} = \overrightarrow{\boldsymbol{v_r}} + \overrightarrow{\boldsymbol{v_e}}$$

Note :

1. When the training motion is translational, the vectors ($\overrightarrow{\imath'}, \overrightarrow{\jmath'} and\ \overrightarrow{k'}$) remain parallel to the unit vectors ($\vec{\imath}, \vec{\jmath}\ and\ \vec{k}$) of the fixed reference frame.

So $\frac{d\overrightarrow{\imath'}}{dt} = \frac{d\overrightarrow{\jmath'}}{dt} = \frac{d\overrightarrow{k'}}{dt} = \vec{0}$ and $\vec{v_e} = \left(\frac{d\overrightarrow{OO'}}{dt}\right)/R$

2. If the moving frame of reference (R') rotates relative to the fixed frame of reference (R), the training velocity can also be written as:

$$\vec{v_e} = \frac{d\overrightarrow{OO'}}{dt} + \vec{\omega}\Lambda\overrightarrow{O'M}$$

With $\vec{\omega}$ represents the rotational velocity of R'/R.

3. In the case of translational motion $\vec{\omega} = \vec{0}$ then $\vec{v_e} = \frac{d\overrightarrow{OO'}}{dt}$.

2.2. Composition of acceleration

Acceleration is the derivative of velocity with respect to time :

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\overrightarrow{OM}}{dt^2} = \frac{d^2x}{dt^2}\vec{\imath} + \frac{d^2y}{dt^2}\vec{\jmath} + \frac{d^2z}{dt^2}\vec{k}$$

$$\vec{v} = \frac{d\overrightarrow{OO'}}{dt} + \frac{\boldsymbol{dx'}}{\boldsymbol{dt}}\overrightarrow{\boldsymbol{\imath'}} + \frac{\boldsymbol{dy'}}{\boldsymbol{dt}}\overrightarrow{\boldsymbol{\jmath'}} + \frac{\boldsymbol{dz'}}{\boldsymbol{dt}}\overrightarrow{\boldsymbol{k'}} + x'\frac{d\overrightarrow{\imath'}}{dt} + y'\frac{d\overrightarrow{\jmath'}}{dt} + z'\frac{d\overrightarrow{k'}}{dt}$$

$$\frac{d}{dt}\left(\frac{dx'}{dt}\overrightarrow{\imath'}\right) = \frac{d^2x'}{dt^2}\overrightarrow{\imath'} + \frac{dx'}{dt}\frac{d\overrightarrow{\imath'}}{dt}$$

$$\frac{d}{dt}\left(\frac{dy'}{dt}\overrightarrow{j'}\right)=\frac{d^2y'}{dt^2}\overrightarrow{j'}+\frac{dy'}{dt}\frac{d\overrightarrow{j'}}{dt}$$

$$\frac{d}{dt}\left(\frac{dz'}{dt}\overrightarrow{k'}\right)=\frac{d^2z'}{dt^2}\overrightarrow{k'}+\frac{dz'}{dt}\frac{d\overrightarrow{k'}}{dt}$$

and

$$\frac{d}{dt}\left(x'\frac{d\overrightarrow{i'}}{dt}\right)=\frac{dx'}{dt}\frac{d\overrightarrow{i'}}{dt}+x'\frac{d^2\overrightarrow{i'}}{dt^2}$$

$$\frac{d}{dt}\left(y'\frac{d\overrightarrow{j'}}{dt}\right)=\frac{dy'}{dt}\frac{d\overrightarrow{j'}}{dt}+y'\frac{d^2\overrightarrow{j'}}{dt^2}$$

$$\frac{d}{dt}\left(z'\frac{d\overrightarrow{k'}}{dt}\right)=\frac{dz'}{dt}\frac{d\overrightarrow{k'}}{dt}+z'\frac{d^2\overrightarrow{k'}}{dt^2}$$

$$\vec{a}=\frac{d^2x'}{dt^2}\overrightarrow{i'}+\frac{d^2y'}{dt^2}\overrightarrow{j'}+\frac{d^2z'}{dt^2}\overrightarrow{k'}+\frac{d^2\overrightarrow{OO'}}{dt^2}+x'\frac{d^2\overrightarrow{i'}}{dt^2}+y'\frac{d^2\overrightarrow{j'}}{dt^2}+z'\frac{d^2\overrightarrow{k'}}{dt^2}+2(\frac{dx'}{dt}\frac{d\overrightarrow{i'}}{dt}+\frac{dy'}{dt}\frac{d\overrightarrow{j'}}{dt}+\frac{dz'}{dt}\frac{d\overrightarrow{k'}}{dt})$$

with

$$\overrightarrow{a_a}=\frac{d^2\overrightarrow{OM}}{dt^2}=\frac{d^2x}{dt^2}\vec{i}+\frac{d^2y}{dt^2}\vec{j}+\frac{d^2z}{dt^2}\vec{k}$$

$$\overrightarrow{a_r}=\frac{d^2\overrightarrow{O'M}}{dt^2}=\frac{d^2x'}{dt^2}\overrightarrow{i'}+\frac{d^2y'}{dt^2}\overrightarrow{j'}+\frac{d^2z'}{dt^2}\overrightarrow{k'}$$

$$\overrightarrow{a_e}=\frac{d^2\overrightarrow{OO'}}{dt^2}+x'\frac{d^2\overrightarrow{i'}}{dt^2}+y'\frac{d^2\overrightarrow{j'}}{dt^2}+z'\frac{d^2\overrightarrow{k'}}{dt^2}$$

$$\Rightarrow\overrightarrow{a_e}=\frac{d^2\overrightarrow{OO'}}{dt^2}+\vec{\omega}\Lambda\left(\vec{\omega}\Lambda\overrightarrow{O'M}\right)+\frac{d\vec{\omega}}{dt}\Lambda\overrightarrow{O'M}$$

$$\overrightarrow{a_c}=\ 2\left(\frac{dx'}{dt}\frac{d\overrightarrow{i'}}{dt}+\frac{dy'}{dt}\frac{d\overrightarrow{j'}}{dt}+\frac{dz'}{dt}\frac{d\overrightarrow{k'}}{dt}\right)$$

$$\Rightarrow\overrightarrow{a_c}=2(\vec{\omega}\Lambda\overrightarrow{v_r})/\mathrm{R'}$$

Then the law of composition of the acceleration will be :

$$\boldsymbol{\overrightarrow{a_a}=\overrightarrow{a_r}+\overrightarrow{a_c}+\overrightarrow{a_e}}$$

$\overrightarrow{\boldsymbol{a_a}}$ is the absolute acceleration representing the second derivative of $\overrightarrow{OM}$ with respect to time in the fixed reference frame. This is the acceleration of M in the fixed frame of reference. par rapport au temps dans le repère fixe.

$\overrightarrow{\boldsymbol{a_r}}$ is the relative acceleration representing the second derivative of $\overrightarrow{O'M}$ with respect to time in the moving frame of reference. This is the acceleration of M in the moving frame of reference.

$\overrightarrow{\boldsymbol{a_e}}$ is the training acceleration, which represents the acceleration of the motion of the moving frame relative to the fixed frame. This is the acceleration of the moving frame R' relative to the fixed frame R.

$\overrightarrow{\boldsymbol{a_c}}$ is the Coriolis or complementary acceleration (it has no physical meaning).

Note:

If the moving frame of reference (R') rotates relative to the fixed frame of reference (R), the drive speed can also be written as:

$$\overrightarrow{v_e} = \left(\frac{d\overrightarrow{OO'}}{dt}\right)/R + (\vec{\omega}\Lambda\overrightarrow{O'M})/R'$$

Training acceleration by:

$$\overrightarrow{a_e} = \left(\frac{d^2\overrightarrow{OO'}}{dt^2}\right)/R + (\frac{d\vec{\omega}}{dt}\Lambda\overrightarrow{O'M})/R' + \left(\vec{\omega}\Lambda\vec{\omega}\Lambda\overrightarrow{O'M}\right)/\text{R'}$$

and the Coriolis acceleration by:

$$\overrightarrow{a_c} = 2.(\vec{\omega}\Lambda\overrightarrow{v_r})/R'$$

With $\vec{\omega}$ represents the rotational speed of R'/R.

Special cases:

1. When the training motion is translational, the vectors $\overrightarrow{\imath'}, \overrightarrow{\jmath'} et \ \overrightarrow{k'}$ remain parallel to the unit vectors ($\vec{\imath}, \vec{\jmath}$ et $\vec{k}$) of the fixed reference frame.

So $\frac{d\overrightarrow{\imath'}}{dt} = \frac{d\overrightarrow{\jmath'}}{dt} = \frac{d\overrightarrow{k'}}{dt} = \vec{0}$ and $\vec{\boldsymbol{\omega}} = \vec{\mathbf{0}}$

Then $\overrightarrow{v_e} = \frac{d\overrightarrow{OO'}}{dt}, \overrightarrow{a_e} = \left(\frac{d^2\overrightarrow{OO'}}{dt^2}\right)$ and $\overrightarrow{a_c} = \vec{0}$.

2. If R' has a uniform rectilinear motion then $\vec{\boldsymbol{\omega}} = \vec{\mathbf{0}}, \overrightarrow{a_c} = \vec{0}$, $\frac{d\overrightarrow{OO'}}{dt} = cst$ and $\frac{d^2\overrightarrow{OO'}}{dt^2} = \vec{0}$ So $\overrightarrow{a_e} = \vec{0}$ because $\overrightarrow{v_e} = cst$.

3. If R' has a pure rotation about R (R' and R have the same origin), so we have:

$\frac{d^2\overrightarrow{OO'}}{dt^2} = \vec{0}$ and $\frac{d\overrightarrow{OO'}}{dt} = \vec{0}$ then $\overrightarrow{v_e} = (\vec{\omega}\Lambda\overrightarrow{O'M})/R'$

And $\overrightarrow{a_e} = (\frac{d\vec{\omega}}{dt}\Lambda\overrightarrow{O'M})/R' + \left(\vec{\omega}\Lambda\vec{\omega}\Lambda\overrightarrow{O'M}\right)$/R'

Application exercise:

The coordinates of a moving particle in the reference frame (R) provided with the reference frame $\left(O, \vec{\imath}, \vec{\jmath}, \vec{k}\right)$ are given as a function of time by:

$$x = 2t^3 + 1, \qquad y = 4t^2 + t - 1, \qquad z = t^2$$

In a second frame of reference (R') with the reference frame $\left(O', \overrightarrow{\imath'}, \overrightarrow{\jmath'}, \overrightarrow{k'}\right)$ with $\vec{\imath} = \overrightarrow{\imath'}, \vec{\jmath} = \overrightarrow{\jmath'}, \vec{k} = \overrightarrow{k'}$ are given by:

$$x' = 2t^3, \qquad y' = 4t^2 - 3t + 2, \qquad z' = t^2 - 5$$

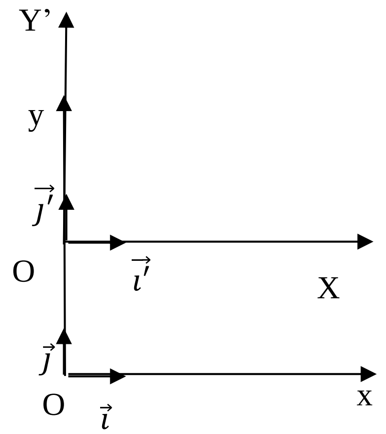
1- Express the velocity v of M in (R) as a function of its velocity v' in (R'), and proceed in the same way for the accelerations.

2- Define the training motion of (R') relative to (R).

Corrected:

$$\overrightarrow{OM}/(R)\begin{cases} x = 2t^3 + 1 \\ y = 4t^2 + t - 1 \\ z = t^2 \end{cases} \quad and \quad \overrightarrow{O'M}(R')\begin{cases} x' = 2t^3 \\ y' = 4t^2 - 3t + 2 \\ z' = t^2 - 5 \end{cases}$$

1. The speed of point M in the fixed reference frame (R) and the moving reference frame (R').



$$\vec{v} = \overrightarrow{v_a} = \frac{d\overrightarrow{OM}}{dt}\begin{cases} \frac{dx}{dt} = 6t^2 \\ \frac{dy}{dt} = 8t + 1 \\ \frac{dz}{dt} = 2t \end{cases}$$

$$\overrightarrow{v'} = \overrightarrow{v_r} = \frac{d\overrightarrow{O'M}}{dt}\begin{cases} \frac{dx'}{dt} = 6t^2 \\ \frac{dy'}{dt} = 8t - 3 \\ \frac{dz'}{dt} = 2t \end{cases}$$

So :

$\vec{v}$= $6t^2\vec{\imath}$ +$(8t + 1)\vec{\jmath} + 2t\vec{k}$ and $\overrightarrow{v'} = 6t^2\vec{\imath}$ +$(8t - 3)\vec{\jmath} + 2t\vec{k}$

We have : $\overrightarrow{v_a} = \overrightarrow{v_r} + \overrightarrow{v_e} \Rightarrow \overrightarrow{v_e} = \overrightarrow{v_a} - \overrightarrow{v_r}$

$\overrightarrow{v_e}$=$(6t^2\vec{\imath}$ +$(8t + 1)\vec{\jmath} + 2t\vec{k}) - (6t^2\vec{\imath}$ +$(8t - 3)\vec{\jmath} + 2t\vec{k})$=$4\vec{\jmath}$

So $\vec{\boldsymbol{v}}$ =$\vec{\boldsymbol{v}}' + \boldsymbol{4}\vec{\boldsymbol{\jmath}}$

And $\vec{\boldsymbol{\imath}} = \vec{\boldsymbol{\imath}'}, \vec{\boldsymbol{\jmath}} = \vec{\boldsymbol{\jmath}'}, \vec{\boldsymbol{k}} = \vec{\boldsymbol{k}'}$

2. The acceleration of point M in the two reference frames, fixed (R) and moving (R'), is as follows:

$$\vec{a} = \overrightarrow{a_a} = \frac{d\vec{v}}{dt} \begin{cases} \dfrac{dv_x}{dt} = 12t \\ \dfrac{dv_y}{dt} = 8 \\ \dfrac{dv_z}{dt} = 2 \end{cases}$$

And $$\overrightarrow{a'} = \overrightarrow{a_r} = \frac{d\overrightarrow{v'}}{dt} \begin{cases} \dfrac{dv'_x}{dt} = 12t \\ \dfrac{dv'_y}{dt} = 8 \\ \dfrac{dv'_z}{dt} = 2 \end{cases}$$

So $\vec{a} = \overrightarrow{a'}$ or $\overrightarrow{a_a}$=$\overrightarrow{a_r}$

Conclusion:

The motion of frame of reference (R') relative to the fixed frame of reference (R) is a uniform translational motion along axis Oy with a constant speed of 4m/s.

Références

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