Chapiter 3

Number Sequences

1st Grade
1st Semester
Calculus 1
Mathematics
Ms RAHMOUN

I/ Vocabulary

متتالية أعداد حقيقية	real number sequence	suite de nombres réels
الحدّ العام	general term	terme général
متزايدة	increasing	croissante
متناقصة	decreasing	décroissante
رتيبة	monotone	monotone
البرهان بالتراجع	Proof by induction	preuve par récurrence
العكس النقيض	The contrapositive	la contraposée
القضية العكسية	the reciprocal	la réciproque
تطبيق	a map	une application
تتقارب - تتباعد	Converges - diverges	Converge - diverge
حالات عدم التعيين	indetemined forms	formes indéterminées
متتاليات مستخرجة	Sub-sequences	sous suites
متتاليات متجاورة	Adjacent sequences	suites adjacente

II/ Generalities

Let E be an non empty set.

We call a Sequence of elements of E any map u:

$$u : \mathbb{N} \longrightarrow E$$
$$n \longmapsto u(n) = u_n$$

1/ The sequence is denoted $(u_n)_{n\in\mathbb{N}}$, $\mathbb{N}$ may be replaced by $\{n_1, n_1+1, \dots$

2/ If $E = \mathbb{R}$, $(u_n)_{n\in\mathbb{N}}$ is then a real number sequence of general term u_n.

3/ ⚠ In all what follows $E = \mathbb{R}$.

Examples

① $(u_n)_{n\geq 1}$ with $u_n = \frac{1}{n}$.

② $(v_n)_{n\geq 0}$ with $v_n = (-1)^n$.

③ $(w_n)_{n\geq 2}$ with $w_n = \sqrt{n^2-2}$.

① Operations on sequences

1°/ Equality: $(U_n)_{n\in\mathbb{N}} = (V_n)_{n\in\mathbb{N}} \iff \forall n\in\mathbb{N}\ U_n = V_n$,

2°/ Sum: $(U_n)_{n\in\mathbb{N}} + (V_n)_{n\in\mathbb{N}} = (W_n)_{n\in\mathbb{N}}$ with:

$\forall n\in\mathbb{N},\ W_n = U_n + V_n$.

3°/ Product: $(U_n)_{n\in\mathbb{N}} \cdot (V_n)_{n\in\mathbb{N}} = (W_n)_{n\in\mathbb{N}}$ with

$\forall n\in\mathbb{N},\ W_n = U_n \cdot V_n$.

4°/ Multiplication by a scalar:

$\forall \lambda\in\mathbb{R},\ \lambda(U_n)_{n\in\mathbb{N}} = (W_n)_{n\in\mathbb{N}}$ with.

$\forall n\in\mathbb{N}\ W_n = \lambda U_n$.

② Properties

1°/ $(U_n)_{n\in\mathbb{N}}$ is upper bounded $\iff \exists M\in\mathbb{R} / U_n \leq M, \forall n\in\mathbb{N}$.

2°/ $(U_n)_{n\in\mathbb{N}}$ is Lower bounded $\iff \exists m\in\mathbb{R} / U_n \geq m, \forall n\in\mathbb{N}$.

3°/ $(U_n)_{n\in\mathbb{N}}$ is bounded $\iff \exists M, m\in\mathbb{R} / m \leq U_n \leq M, \forall n\in\mathbb{N}$.

or $\iff \exists \alpha\in\mathbb{R}^+ / |U_n| \leq \alpha, \forall n\in\mathbb{N}$.

4°/ $(U_n)_{n\in\mathbb{N}}$ is increasing $\iff \forall n\in\mathbb{N},\ U_n \leq U_{n+1}$, (we write $(U_n)_{n\in\mathbb{N}} \nearrow$)

—— strictly increasing $\iff$ ——, $U_n < U_{n+1}$.

5°/ $(U_n)_{n\in\mathbb{N}}$ is Constant $\iff \forall n\in\mathbb{N}\ U_n = U_{n+1}$

6°/ $(U_n)_{n\in\mathbb{N}}$ is decreasing $\iff \forall n\in\mathbb{N},\ U_n \geq U_{n+1}$, (we write $(U_n)_{n\in\mathbb{N}} \searrow$)

—— strictly decreasing $\iff \forall n\in\mathbb{N}\ U_n > U_{n+1}$

7°/ $(U_n)_{n\in\mathbb{N}}$ is monotone $\iff (U_n)_{n\in\mathbb{N}} \nearrow$ or $\searrow$.

—— stricty monotone $\iff (U_n)_{n\in\mathbb{N}}$ is stric. $\nearrow$ or stric. $\searrow$

③ Theorem 1

Let be $(U_n)_{n\in\mathbb{N}} / \forall n\in\mathbb{N}\ U_n > 0$

1/ $(U_n)_{n\in\mathbb{N}}$ is increasing $\iff \dfrac{U_{n+1}}{U_n} \geq 1$, $\forall n\in\mathbb{N}$ (strict $>$).

2/ $(U_n)_{n\in\mathbb{N}}$ is decreasing $\iff \dfrac{U_{n+1}}{U_n} \leq 1$ $\forall n\in\mathbb{N}$ (strict. $<$).

Example show, at home, that the sequence $(U_n)_{n\in\mathbb{N}^*}$ defined by the general term $U_n = \dfrac{n}{n+1}$ is strictly increasing, and bounded.

III Convergent sequences

① **Definition** Let $(U_n)_{n\in\mathbb{N}}$ a number sequence and $\ell \in \mathbb{R}$.

We say that $(U_n)_{n\in\mathbb{N}}$ converges to ℓ (or toward ℓ) iff $\boxed{\lim_{n\to+\infty} U_n = \ell}$, we write $U_n \xrightarrow[n\to+\infty]{} \ell$ or $U_n \to \ell$.

This is equivalent to have:

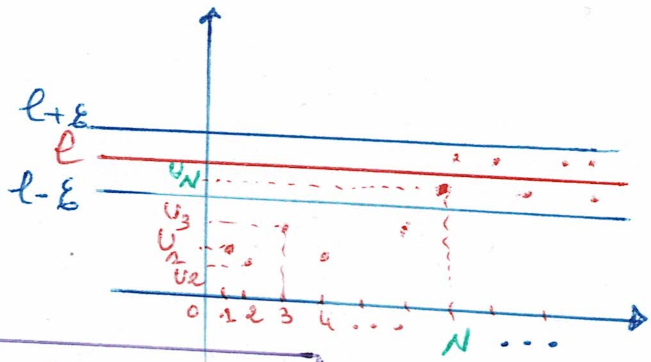
$$\boxed{\forall \varepsilon > 0, \exists N \in \mathbb{N} / \forall n, n \geq N \Rightarrow |U_n - \ell| < \varepsilon}$$

1°) So, to show that $(U_n)_{n\in\mathbb{N}}$ converges to ℓ, we have to show that for every ε, as smal as we want, there is a rank $N \in \mathbb{N}$ from where (i.e. $\forall n \geq N$), we get $\ell - \varepsilon < U_n < \ell + \varepsilon$.

2°) It must be pointed out that N is not unique, indeed, for every $n_1 \geq N$, the inequality $|U_n - \ell| < \varepsilon$ is verified. So n_1 may play the role of rank. That is why.

$$(U_n)_{n\geq 0} \to \ell \iff (U_n)_{n \geq n_1} \to \ell .$$

3°) In fact, N depends upon ε.



② **Theorem**

$$\boxed{\text{If } (U_n)_{n\in\mathbb{N}} \longrightarrow \ell, \text{ then } \ell \text{ is unique.}}$$

Proof By the absurd, suppose that $(U_n)_{n\in\mathbb{N}} \longrightarrow \ell'$ and $\ell \neq \ell'$.

Well, $\ell \neq \ell' \Rightarrow \ell > \ell'$ or $\ell < \ell'$

Suppose that $\ell > \ell'$

$$\ell > \ell' \Rightarrow \frac{\ell - \ell'}{2} > 0 .$$

So, take $\varepsilon = \dfrac{\ell - \ell'}{2} > 0$. Therefore, $\exists N_1 \in \mathbb{N}, \forall n, n \geq N_1 \Rightarrow |U_n - \ell| < \dfrac{\ell - \ell'}{2}$

i.e. $\exists N_1 \in \mathbb{N} / \forall n, n \geq N_1 \Rightarrow \dfrac{\ell + \ell'}{2} < U_n < \dfrac{3\ell - \ell'}{2}$ ... (1)

By the same way, since $(U_n)_{n\in\mathbb{N}} \longrightarrow \ell'$, we get: $\exists N_2 \in \mathbb{N} /$

$\forall n, \ n \geq N_2 \Rightarrow \dfrac{3\ell' - \ell}{2} < U_n < \dfrac{\ell + \ell'}{2}$ ... (2)

So, for $n \geq \max(N_1, N_2)$ we get

$$\begin{cases} \frac{\ell+\ell'}{2} < U_n & \text{from } ① \\ U_n < \frac{\ell+\ell'}{2} & \text{from } ② \end{cases}$$

which is absurd.

Same thing when $\ell < \ell'$.

Conclusion $\ell = \ell'$.

Examples: Using the definition, show that the following sequences converges to ℓ.

① $(U_n)_{n\in\mathbb{N}^*}$ / $U_n = \frac{1}{n}$, $\ell = 0$.

② $(U_n)_{n\in\mathbb{N}^*}$ / $U_n = \frac{(-1)^n}{n}$, $\ell = 0$.

③ $(U_n)_{n\in\mathbb{N}^*}$ / $U_n = \frac{(2n^2+1)^2}{n^4}$, $\ell = 4$.

④ $(U_n)_{n\in\mathbb{N}}$ / $U_n = a$, $(a\in\mathbb{R})$; $\ell = a$.

⑤ $(U_n)_{n\in\mathbb{N}^*}$ / $U_n = \sqrt[n]{a}$, $(a > 1)$; $\ell = 1$.

Correction

We will use the same definition every time:

$$\boxed{\forall \varepsilon > 0, \exists N \in \mathbb{N} / \forall n, n \geq N \Rightarrow |U_n - \ell| < \varepsilon}$$

① Let be $\varepsilon > 0$, find N that verifies,

$$n \geq N \Rightarrow |U_n - 0| < \varepsilon$$

$$|U_n - 0| = \left|\frac{1}{n} - 0\right| = \frac{1}{n}$$

to get $\frac{1}{n} < \varepsilon$ we must have $n > \frac{1}{\varepsilon}$

$\mathbb{R}$ been archimedean, $\forall \varepsilon > 0, \exists N \in \mathbb{N} / N > \frac{1}{\varepsilon}$

we may choose $N = E(\frac{1}{\varepsilon}) + 1$.

②

$$|U_n - 0| = \left|\frac{(-1)^n}{n} - 0\right| = \frac{1}{n} \quad \text{So same as in ①.}$$

③ $$|U_n - 4| = \left|\frac{(2n^2+1)^2}{n^4} - 4\right| = \left|\frac{4n^2+1}{n^4}\right| = \frac{4}{n^2} + \frac{1}{n^4}$$

We want to find $N \in \mathbb{N} / \forall n, n \geq N \Rightarrow \frac{4}{n^2} + \frac{1}{n^4} < \varepsilon$.

One has : $\forall n \in \mathbb{N}^*$ $n^2 \leq n^4 \Rightarrow \frac{1}{n^2} \geq \frac{1}{n^4}$

$$\Rightarrow \frac{4}{n^2} + \frac{1}{n^2} \geq \frac{4}{n^2} + \frac{1}{n^4}$$

$$\Rightarrow \frac{4}{n^2} + \frac{1}{n^4} \leq \frac{5}{n^2}$$

To ensure that $\frac{4}{n^2} + \frac{1}{n^4} < \varepsilon$, we may impose that $\frac{5}{n^2} < \varepsilon$.

This is always possible by taking $N = E\left(\sqrt{\frac{5}{\varepsilon}}\right) + 1$.

Because, to have $\frac{5}{n^2} < \varepsilon$, we must have $n^2 > \frac{5}{\varepsilon}$, or $n > \sqrt{\frac{5}{\varepsilon}}$.

④ $|U_n - a| = |a - a| = 0 < \varepsilon$, $\forall \varepsilon > 0$

So any $n \in \mathbb{N}$ will verify $|U_n - a| < \varepsilon$, we may take $N = 0$.

⑤ $|U_n - 1| = |\sqrt[n]{a} - 1| = \sqrt[n]{a} - 1$; (because $a > 1$)

Solve $\sqrt[n]{a} - 1 < \varepsilon$.

$$\Rightarrow \sqrt[n]{a} < \varepsilon + 1 \Rightarrow a^{\frac{1}{n}} < (\varepsilon + 1)$$

⚠ $a > 1 \Rightarrow \ln a > 0$

and, $\varepsilon + 1 > 1 \Rightarrow \ln(\varepsilon + 1) > 0$

$$\Rightarrow \frac{1}{n} \ln a < \ln(\varepsilon + 1) \Rightarrow n > \frac{\ln a}{\ln(\varepsilon + 1)}.$$

It is sufficient to take $N = E\left(\frac{\ln a}{\ln(\varepsilon + 1)}\right) + 1$.

③ Definitions:

1/ $(U_n)_{n\in\mathbb{N}}$ is convergent $\Leftrightarrow \exists \ell \in \mathbb{R} / (U_n)_{n\in\mathbb{N}} \longrightarrow \ell$.

2/ A sequence that doesn't converge is said to be divergent. This includes the following cases:

$(U_n)_{n\in\mathbb{N}} \longrightarrow A$ where $\boxed{A = \pm\infty}$ First type divergence

or, Second type divergence: A does not exist (such as : multiple plausible values of A).

3/ Studying a sequence, or studying the nature of a sequence is finding out if it is convergent or divergent this means that we have to study the $\lim\limits_{n \to +\infty} U_n$.

④ Corollary:

1/ If $U_n \longrightarrow \ell$ and $\ell > 0$, then $\exists n_0 \in \mathbb{N} / \forall n \geq n_0, U_n > 0$.

2/ If $U_n \longrightarrow \ell$ and $\ell < 0$, then $\exists n_0 \in \mathbb{N} / \forall n \geq n_0, U_n < 0$.

Proof: $(U_n)_{n\in\mathbb{N}} \rightarrow \ell \Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N} / \forall n, n \geq N \Rightarrow |U_n - \ell| < \varepsilon$

So, $n \geq N \Rightarrow \ell - \varepsilon < U_n < \ell + \varepsilon$

Suppose $\ell > 0$ and choose $\varepsilon = \ell$, we then find
$N \in \mathbb{N} / \forall n,\ n \geq N \Rightarrow \quad \ell - \ell < U_n < \ell + \ell$

$$\Rightarrow \quad \underbrace{0 < U_n}_{} < 2\ell$$

$$n \geq N \Rightarrow \quad 0 < U_n$$

Considering $n_0 = N$ achieves the proof.

Same work when $\ell < 0$. (to do at home).

Example. Show that the sequence defined by the general term $U_n = (-1)^n$ diverges.

By the absurd, suppose that $\exists! \ell \in \mathbb{R} / U_n \longrightarrow \ell$.

① if $\ell > 0$ then there exist a precise rank, from which all terms of the sequence are positive, which is absurd.

② if $\ell < 0$, then all terms of the sequence are now strictly negative going from a certain rank, which is absurd.

③ if $\ell = 0$, then $\forall \varepsilon > 0, \exists N \in \mathbb{N} / \forall n,\ n \geq N \Rightarrow |U_n - 0| < \varepsilon$.

$$|U_n - 0| = |(-1)^n| = 1 < \varepsilon$$

We are saying that $\forall \varepsilon > 0 \ \ldots \ 1 < \varepsilon$! which is absurd.

In all cases, our supposition is false. This means that $(U_n)_{n \in \mathbb{N}} / U_n = (-1)^n$ diverges.

⑤ **Theorem 3** $\boxed{\text{Every convergent sequence is bounded.}}$

Proof

Let $(U_n)_{n \in \mathbb{N}}$ a sequence that converges to ℓ. ($\ell \in \mathbb{R}$).

→ Taking $\varepsilon = 1$, one gets:

$$\exists N \in \mathbb{N} / \forall n,\ n \geq N \Rightarrow |U_n - \ell| < 1$$
$$\Rightarrow \ell - 1 < U_n < \ell + 1$$

→ Considering that, when $\underset{\text{finite number of terms}}{n \leq N-1}$ $U_n \leq U_n \leq U_n$

Then $\min(\ell - 1, U_{N-1}, U_{N-2}, \ldots, U_0) \leq U_n \leq \max(\ell + 1, U_{N-1}, \ldots, U_0)$

→ Put: $\alpha = \min(\ell - 1, U_{N-1}, \ldots, U_0)$ and $\beta = \max(\ell + 1, U_{N-1}, \ldots, U_0)$

Now, $\forall n \in \mathbb{N}$ $\alpha \leq U_n \leq \beta$, which compleats the proof.

Remarks

1/ theorem 3 says that: $(U_n)_{n\in\mathbb{N}}$ cvg $\Rightarrow$ $(U_n)_{n\in\mathbb{N}}$ is bounded.
the reciprocal is not always true. For instance,
The sequence $(-1)^n$ is bounded yet divergent.

2/ We sometimes, use the contrapositive of theorem 3, that is that: $(U_n)_{n\in\mathbb{N}}$ is not bounded $\Rightarrow (U_n)_{n\in\mathbb{N}}$ is divergent.

As a simple example, Consider $(U_n)_{n\in\mathbb{N}}$ / $U_n = n$.

6 Operations and Convergence

Let $(U_n)_{n\in\mathbb{N}}$ and $(V_n)_{n\in\mathbb{N}}$ two sequences that converge to ℓ and ℓ' respectively. Then One has:

1./ $(U_n + V_n)_{n\in\mathbb{N}} \longrightarrow \ell + \ell'$.

2./ $(U_n \cdot V_n)_{n\in\mathbb{N}} \longrightarrow \ell \cdot \ell'$.

3./ $\forall \lambda \in \mathbb{R}$, $(\lambda U_n)_{n\in\mathbb{N}} \longrightarrow \lambda \ell$.

4./ If $\ell' \neq 0$ i.e $\exists N \in \mathbb{N} / \forall n \geq N$ $V_n \neq 0$ then,

$$\left(\frac{U_n}{V_n}\right)_{n\in\mathbb{N}} \longrightarrow \frac{\ell}{\ell'}.$$

⚠ Beware from the division. When $\ell' = 0$, we may get different results, for example: $\lim\limits_{n\to+\infty} \frac{1}{n} = 0$; $\lim\limits_{n\to+\infty} \frac{1}{n+2} = 0$, but

$$\lim_{n\to+\infty} \frac{\frac{1}{n}}{\frac{1}{n+2}} = \lim_{n\to+\infty} \frac{n+2}{n} = 1 \,!$$

5./ If, $\forall n \in \mathbb{N}$ $U_n \leq V_n$ then $\ell \leq \ell'$.

6./ If $\forall n \in \mathbb{N}$ $U_n < V_n$ then $\ell \leq \ell'$.

7 Some Convergence theorems,

Theorem 4:

Let $(U_n)_{n\in\mathbb{N}}$ and $(V_n)_{n\in\mathbb{N}}$ two number sequences.

1./ If $(U_n)_{n\in\mathbb{N}} \to \ell$, then $(|U_n|)_{n\in\mathbb{N}} \to \ell$. $(\ell \in \mathbb{R})$.

2./ If $(|U_n|)_{n\in\mathbb{N}} \longrightarrow 0$, then $(U_n)_{n\in\mathbb{N}} \longrightarrow 0$.

3./ If $(U_n)_{n\in\mathbb{N}} \longrightarrow 0$ and $(V_n)_{n\in\mathbb{N}}$ is bounded, then $(U_n \cdot V_n)_{n\in\mathbb{N}} \longrightarrow 0$.

Proof

1°/ $(U_n)_{n\in\mathbb{N}} \to \ell \iff \forall \varepsilon>0, \exists N\in\mathbb{N} / \forall n,$

$n \geq N \Rightarrow |U_n - \ell| < \varepsilon$

But $\forall n\in\mathbb{N}$. $\big||U_n| - |\ell|\big| \leq |U_n - \ell|$

So $\forall \varepsilon>0, \exists N\in\mathbb{N} / \forall n, \big||U_n| - |\ell|\big| < \varepsilon$ ■

2°/ $(|U_n|)_{n\in\mathbb{N}} \to 0 \iff \forall\varepsilon>0, \exists N\in\mathbb{N} / \forall n$

$n \geq N \Rightarrow \big||U_n| - 0\big| < \varepsilon$

$\Rightarrow |U_n| < \varepsilon$

or

$\Rightarrow |U_n - 0| < \varepsilon$, for the same rank N. ■

3°/ We want to proove that:

$\forall\varepsilon>0, \exists N\in\mathbb{N} / \forall n, n\geq N \Rightarrow |U_n \cdot V_n| < \varepsilon$

we know that

$\forall \varepsilon>0, \exists N'\in\mathbb{N} / \forall n, n \geq N' \; |U_n| < \varepsilon \; \left((U_n)_{n\in\mathbb{N}} \to 0\right)$

and

$\exists M>0 / \forall n, |V_n| \leq M$. $\left((V_n)_{n\in\mathbb{N}}\text{ is bounded}\right)$

Let $\varepsilon>0$. One has $\forall n, |U_n \cdot V_n| = |U_n| \cdot |V_n| \leq M|U_n|$

So Considering $\varepsilon' = \dfrac{\varepsilon}{M} > 0$

$\exists N'\in\mathbb{N} / \forall n, n\geq N' \Rightarrow |U_n| < \varepsilon'$

$\Rightarrow |U_n| \cdot |V_n| < \varepsilon' |V_n|$

$\Rightarrow |U_n \cdot V_n| < \dfrac{\varepsilon}{M} \cdot M$

i.e $\exists N = N' / \forall n, n\geq N \Rightarrow |U_n \cdot V_n| < \varepsilon$. ■

Example What is the nature of the sequence defined by the general term: $U_n = \dfrac{\cos n}{n+2}$

Well, since $\forall n\in\mathbb{N}$ $|\cos n| \leq 1$ (Cos is bounded)

and $\left(\dfrac{1}{n+2}\right)_{n\in\mathbb{N}} \to 0$, then $(U_n)_{n\in\mathbb{N}} \to 0$.

Theorem 5 : Monotony and Convergence

1. Every upper bounded increasing sequence converges to it supremum.
2. Every lower bounded decreasing sequence converges to it infimum.

Proof: Let $(U_n)_{n\in\mathbb{N}}$ an upper bounded and increasing sequence.

Let us show that $(U_n)_{n\in\mathbb{N}}$ is convergent

Put $E=\{U_n, n\in\mathbb{N}\}$ then $\begin{cases} E\neq\emptyset & \text{(otherwise, there is no sequence)} \\ E \text{ is upper bounded.} \end{cases}$

$\Rightarrow$ $\sup E$ exists.

Put $M=\sup E$. Let $\varepsilon>0$; $\overset{?}{\exists} N'\in\mathbb{N} / \forall n, n\geq N' \Rightarrow |U_n - M| < \varepsilon$

One has $M=\sup \iff \exists N\in\mathbb{N} / M-\varepsilon<U_N\leq M$ ... (1)

(Characterization of sup)

Since $(U_n)_{n\in\mathbb{N}}$ is increasing, then, $\forall n / n\geq N$, $U_n\geq U_N$

So, we have

$$\forall n, \; n\geq N \Rightarrow U_n\geq U_N > M-\varepsilon \quad \text{(from (1))}.$$

$$\Rightarrow M-\varepsilon<U_n$$

$$\Rightarrow M-\varepsilon<U_n\leq M \quad (\text{because } M=\sup E).$$

$$\Rightarrow M-\varepsilon<U_n<M+\varepsilon \quad (\text{because } M<M+\varepsilon)$$

It is sufficient to take $N'=N$ to finish the proof.

Same way to proove 2. (to do at home).

Example: Consider the sequence $(U_n)_{n\in\mathbb{N}^*}$ defined by:

$$U_n=\sum_{k=1}^{m}\frac{1}{n+k}=\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{n+m}.$$

Write the three first terms of U_n, then discuss its nature.

Correction:

$$U_1=\sum_{k=1}^{1}\frac{1}{1+k}=\frac{1}{1+1}=\frac{1}{2}.$$

$$U_2=\sum_{k=1}^{2}\frac{1}{2+k}=\frac{1}{2+1}+\frac{1}{2+2}=\frac{1}{3}+\frac{1}{4}=\frac{7}{12}$$

$$U_3=\sum_{k=1}^{3}\frac{1}{3+k}=\frac{1}{3+1}+\frac{1}{3+2}+\frac{1}{3+3}=\frac{1}{4}+\frac{1}{5}+\frac{1}{9}.$$

•/ Finding the monotony:

$$U_{n+1}-U_n = \sum_{k=1}^{n+1}\frac{1}{n+1+k} - \sum_{k=1}^{n}\frac{1}{n+k}$$

$$= \left(\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{2n+2}\right) - \left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots\frac{1}{n+n}\right)$$

$$\cdots\frac{1}{2n}+\frac{1}{2n+1}$$

$$= \frac{1}{2n+1}+\frac{1}{2n+2}-\frac{1}{n+1} = \frac{2(n+1)+(2n+2)-2(2n+1)}{2(n+1)(2n+1)}$$

$$= \frac{1}{2(n+1)(2n+1)} > 0, \quad \forall n\in\mathbb{N}^*$$

So, $(U_n)_{n\in\mathbb{N}}$ is strictely increasing ... (1)

•/ Showing that $(U_n)_{n\in\mathbb{N}}$ is upper bounded:

One has $\forall n\in\mathbb{N}^*$,

$$\left.\begin{array}{l} n+1\geq n+1 \\ n+2\geq n+1 \\ \vdots \\ n+n\geq n+1 \end{array}\right\} \begin{array}{l} \Rightarrow \frac{1}{n+1}\leq\frac{1}{n+1} \\ \Rightarrow \frac{1}{n+2}\leq\frac{1}{n+1} \\ \quad\vdots \\ \Rightarrow \frac{1}{n+n}\leq\frac{1}{n+1} \end{array} \Bigg\} \Rightarrow \sum_{k=1}^{n}\frac{1}{n+1}\leq\sum_{k=1}^{n}\left(\frac{1}{n+1}\right)$$

$\Rightarrow \forall n\in\mathbb{N}^*, \; U_n \leq \frac{n}{n+1} < 1$ (becaus $n<n+1$)

So $(U_n)_{n\in\mathbb{N}}$ is upper bounded by 1 ... (2)

From (1) and (2); $(U_n)_{n\in\mathbb{N}}$ is a convergent sequence.

Theorem 6 The three sequences theorem

Let be $(U_n)_n$, $(V_n)_n$ and $(W_n)_n$ three sequences verifing:

$\exists n_1\in\mathbb{N} \,/\, \forall n\geq n_1 \quad V_n\leq U_n\leq W_n$

If $(V_n)_n$ and $(W_n)_n$ converge to the same limit ℓ, then (U_n) does as well. (i.e. $\begin{cases} v_n\to\ell \\ w_n\to\ell\end{cases} \Rightarrow U_n\to\ell$).

Proof Let be $\varepsilon>0$, we are looking for an N such that

$\forall n, \quad n\geq N \Rightarrow |U_n-\ell|<\varepsilon$

for this ε one has:

$\exists N_1\in\mathbb{N} \,/\, \forall n, \quad n\geq N_1 \Rightarrow \ell-\varepsilon < v_n < \ell+\varepsilon$

and $\exists N_2\in\mathbb{N} \,/\, \forall n, \quad n\geq N_2 \Rightarrow \ell-\varepsilon < w_n < \ell+\varepsilon$

Considering $N=\max(N_1, N_2, n_1)$ we get:

$$\forall n,\quad n \geq N \Rightarrow \begin{cases} \ell - \varepsilon < v_n \\ w_n < \ell + \varepsilon \\ v_n \leq U_n \leq w_n . \end{cases}$$

In other words:

$$\forall n,\quad n \geq N \Rightarrow \quad \ell - \varepsilon < v \leq U_n \leq w_n < \ell + \varepsilon$$

Example.

Study the nature of $(U_n)_{n\in\mathbb{N}^*}$ /

$$U_n = \frac{E(na)}{n} \quad ; \quad a \in \mathbb{R}.$$

One has

$$E(na) \leq na < E(na) + 1$$

$$\Rightarrow \quad na - 1 < E(na) \leq na$$

$$\Rightarrow \quad \underbrace{a - \frac{1}{n}}_{\to a} < \frac{E(na)}{n} \leq \underbrace{a}_{\to a}$$

i.e. $(U_n)_{n\in\mathbb{N}^*} \longrightarrow a.$

IV/ Sub sequences

Definition: Let be $(U_n)_{n\in\mathbb{N}}$ a number sequence and $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$ a strictely increasing map. $\quad \forall n \in \mathbb{N} \;\; \varphi(n) \geq n.$

The sequence defined by $\forall n \in \mathbb{N},\; v_n = U_{\varphi(n)}$ is called a subsequence of $(U_n)_{n\in\mathbb{N}}$.

Examples ① $\varphi_1(n) = 2n$ (strictely increasing map), $\varphi_2(n) = 2n+1.$

$$U_n \to U_0 \quad U_1 \quad U_2 \quad U_3 \quad U_4 \cdots \quad U_{2k} \quad U_{2k+1} \cdots$$

$$v_n = U_{2n} \to v_0 \quad v_1 \quad v_2 \cdots v_k \cdots$$

$$w_n = U_{2n+1} \to w_0 \quad w_1 \cdots \quad w_k \cdots$$

$(v_n)_n = (U_{2n})_n$ and $(U_{2n+1})_n = (w_n)_n$ are subsequences of $(U_n)_{n\in\mathbb{N}}$.

② $\varphi(n) = \begin{cases} n & \text{if } n \leq 5 \\ 7 & \text{if } n \geq 6 \end{cases}$ is Not strictely increasing

$$E = \{\varphi(n) \,/\, n \in \mathbb{N}\} = \{0, 1, 2, 3, 4, 5, 7\}$$

This means that $(v_n)_{n\in\mathbb{N}}$ / $v_n = U_{\varphi(n)}$ is not a subsequence of $(U_n)_{n\in\mathbb{N}}$

$U_n \longrightarrow \quad U_0 \; U_1 \dots U_5 \quad U_6 \quad U_7 \; U_8 \dots$

$v_n = U_{\varphi(n)} \longrightarrow v_0 \; v_1 \dots v_5 \qquad \underbrace{v_6 = v_7 = \dots}$ Constant = not increasing strictely.

③ $\varphi_1(n) = 2n$ et $\varphi_2(n) = 2n+1$; $(U_n) = (-1)^n$

So, $\left.\begin{array}{l} U_{2n} = 1 \\ U_{2n+1} = -1 \end{array}\right\} \forall n \in \mathbb{N}$

if $\varphi_3(n) = 3n$ then $U_{3n} = (-1)^{3n} = \left((-1)^3\right)^n = (-1)^n$

So $U_{3n} = U_n \quad (\forall n \in \mathbb{N})$.

Theorem 7

Let $(U_n)_n$ be a number sequence.
If $(U_n)_n$ converges to l, then every sub-sequence of $(U_n)_n$ converges to l too.

Proof: Let $(U_n)_n$ be a sequence that converges to l.
Put $v_n = U_{\varphi(n)}$, where φ is a strictly increasing map.
We want to proove that v_n converges to l also.
Let be $\varepsilon > 0$, $\exists ? N_0 \in \mathbb{N} / \forall n \geq N_0$, $|v_n - l| < \varepsilon$. ?
For this ε, $\exists N_1 \in \mathbb{N} / \forall n \geq N \Rightarrow |U_n - l| < \varepsilon$

but $\forall n \geq N_1 \quad \varphi(n) \geq n \geq n_1$

$\Rightarrow \varphi(n) \geq n_1$

$\Rightarrow |U_{\varphi(n)} - l| < \varepsilon$.

$\Rightarrow |v_n - l| < \varepsilon$.

Taking $N_0 = N_1$ finishes the proof.

Remark: We often use the Contrapositive of Theorem 6 to Proove that a sequence is divergent, as follows:
- Finding a subsequence that is divergent means that the sequence diverges too.
- Or finding two different sub-sequences that converge to two different limits. Example $U_n = (-1)^n$.

Theorem 8

1/ If $(U_{2n})_n$ and $(U_{2n+1})_n$ converge to the same limit ℓ, then the sequence $(U_n)_n$ converges to ℓ too. (See proof in TD).

2/ The Bolzano - Weirstrass property: from any bounded sequence, we can extract a convergent sub-sequence.

Proof: admited.

V / Cauchy Sequences

① Definition: a sequence $(U_n)_n$ is said to be a Cauchy sequence iff:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \,/\, \forall p, q \in \mathbb{N} \quad p \geq q \geq N \Rightarrow |U_p - U_q| < \varepsilon$$

Not Cauchy $\Leftrightarrow$ $\boxed{\exists \varepsilon > 0 \,/\, \forall N \in \mathbb{N},\ \exists p, q \in \mathbb{N} \quad p \geq q \geq N \wedge |U_p - U_q| \geq \varepsilon}$

② Examples

1/ The Geometric sequence $(U_n)_{n\in\mathbb{N}}$ defined by its general term $U_n = K^n$, $0 < K < 1$ is a Cauchy sequence. Indeed,

Let $\varepsilon > 0$,

$$\exists ? N \in \mathbb{N} \,/\, \forall p, q \in \mathbb{N}, \quad p \geq q \geq N \Rightarrow |U_p - U_q| < \varepsilon \ ?$$

$$|U_p - U_q| = |K^p - K^q| \qquad \boxed{p \geq q \Rightarrow p = q + n,\ n \in \mathbb{N}}$$

$$= K^q |K^n - 1| \leq K^q \qquad |K^n - 1| < 1$$

One has $q \geq N \Rightarrow K^q \leq K^N$ (because $0 < K < 1$)

So, to get $K^N < \varepsilon \Rightarrow N \ln K < \ln \varepsilon$

$\Rightarrow N > \frac{\ln \varepsilon}{\ln K}$, $(\ln K < 0)$

$0 < K < 1$
$\Rightarrow 0 < K^n < 1$
$\Rightarrow -1 < K^n - 1 < 0$
$\Rightarrow |K^n - 1| < 1$

taking $\boxed{N = E\left(\frac{\ln \varepsilon}{\ln K}\right) + 1}$ we get, $\forall p \geq q \geq N$, $|U_p - U_q| < \varepsilon$.

2/ $(U_n)_{n\in\mathbb{N}^*} \,/\, U_n = \frac{1}{n}$ is a Cauchy sequence, because

Let $\varepsilon > 0$, we are looking for $N \in \mathbb{N}$ such that

$$\forall p \geq q \geq N, \text{ we get } |U_p - U_q| < \varepsilon$$

$|U_p - U_q| = \left|\frac{1}{p} - \frac{1}{q}\right| = \left|\frac{1}{q+n} - \frac{1}{q}\right| = \frac{n}{q(q+n)}$, and since $\frac{n}{n+q} < 1$

then $|U_p - U_q| < \frac{1}{q}$

$$q \geq N \Rightarrow \frac{1}{q} \leq \frac{1}{N}$$

So to get $\frac{1}{q} \leq \frac{1}{N} < \varepsilon$ It is sufficient to take

$$\boxed{N = E\left(\frac{1}{\varepsilon}\right) + 1}$$

3./ We can demonstrate that the sequence defined by its general term: $U_n = \frac{1}{n} + (-1)^n$ is not a Cauchy one. $(n \in \mathbb{N}^*)$

One has $|U_p - U_q| = \left|\frac{1}{p} + (-1)^p - \frac{1}{q} - (-1)^q\right|$

taking $p = 2N+2$ and $q = 2N+1$, One gets

$$|U_{2N+2} - U_{2N+1}| = \left|\frac{1}{2N+2} + (-1)^{2N+2} - \frac{1}{2N+1} - (-1)^{2N+1}\right| = \left|2 - \frac{1}{(2N+2)(2N+1)}\right| \geq 1$$

(because $\forall k \in \mathbb{N}^*$ $\frac{1}{k} \leq 1$ $\Rightarrow \left(2 - \frac{1}{k}\right) \geq 1$)

So, $\exists \varepsilon = 1 > 0 \,/\, \forall N \in \mathbb{N}$, $\exists p = 2N+2$, and $q = 2N+1$ such that

$$p \geq q \geq N \wedge |U_p - U_q| \geq \varepsilon.$$

4./ Show that $(U_n)_{n \in \mathbb{N}} \,/\, U_n = \sqrt{n}$ is not a Cauchy sequence.

$$|U_p - U_q| = |\sqrt{p} - \sqrt{q}|$$

taking $p = 4(N+1)$ and $q = N+1$ One find that

$$|U_p - U_q| = \left|\sqrt{4(N+1)} - \sqrt{N+1}\right| = \sqrt{N+1} \geq 1, \forall N \in \mathbb{N}$$

So $\exists \varepsilon = 1 > 0, / \forall N \in \mathbb{N}, \exists p = 4(N+1), \exists q = (N+1) /$

$$p \geq q \geq N \wedge |U_p - U_q| \geq 1$$

5./ Show that $(U_n)_{n \in \mathbb{N}^*} \,/\, U_n = \ln n$ is not a Cauchy sequence.

$|U_p - U_q| = |\ln p - \ln q|$. Taking $p = 2(N+1)$, $q = N+1$, $N \in \mathbb{N}$

On finds out that:

$$|U_p - U_q| = |\ln(2(N+1)) - \ln(N+1)| = \left|\ln \frac{2(N+1)}{N+1}\right| = \underbrace{\ln 2}_{= \varepsilon > 0}$$

So $\exists \varepsilon = \ln 2 > 0 / \forall N \in \mathbb{N}, \exists p = 2(N+1), \exists q = (N+1) /$

$$p \geq q \geq N \wedge |U_p - U_q| = \varepsilon.$$

Exercise Show that :

1./ $(U_n)_{n\in\mathbb{N}^*}$ defined by $U_n = \sum_{k=1}^{n}\left(\frac{1}{k}\right)$ is not a Cauchy sequence

2/ $(U_n)_{n\in\mathbb{N}^*}$ defined by $U_n = \sum_{k=1}^{n}\left(\frac{1}{k^2}\right)$ is a Cauchy sequence

Theorem 9

1./ Every Cauchy sequence is bounded.

2./ $(U_n)_n$ is Cauchy $\iff$ $(U_n)_n$ is convergent

Proof admitted.

VI/ Extention to infinite limits

① Definition

⊙ A sequence $(U_n)_n$ tends to $+\infty$ iff $\forall A>0, \exists N\in\mathbb{N} / n\geq N \Rightarrow U_n > A$

We write $\lim U_n = +\infty$. or $\lim\limits_{n\to+\infty} U_n = +\infty$ or $U_n \longrightarrow +\infty$

⊙ A sequence $(U_n)_n$ tends to $-\infty$ iff $\forall A>0, \exists N\in\mathbb{N} / n\geq N \Rightarrow U_n < -A$

We write $\lim U_n = -\infty$ or $\lim\limits_{n\to+\infty} U_n = -\infty$. or $U_n \longrightarrow -\infty$

⊙ when $\lim U_n = \pm\infty$ we say that $(U_n)_{n\in\mathbb{N}}$ diverges and that this divergence is of the first kind.

And when the limite of (U_n) does not exist, we say that this is a divergence of the second kind. Example : $(U_n)_n / U_n = (-1)^n$.

Theorem 10

Let $(U_n)_n$ and $(V_n)_n$ be two number sequences.

1/ If $(U_n)_n$ is increasing and non-upper bounded then it diverges to $+\infty$

2/ If $(U_n)_n$ is decreasing and non-lower bounded then it diverges to $-\infty$

3/ If $\exists n_1 \in\mathbb{N} / \forall n\geq n_1$ $U_n \leq V_n$ and $U_n \longrightarrow +\infty$ then $V_n \to +\infty$

4/ If $\exists n_1 \in\mathbb{N} / \forall n\geq n_1$ $U_n \leq V_n$ and $V_n \longrightarrow -\infty$ then $U_n \longrightarrow -\infty$

5/ If $U_n \longrightarrow +\infty$ then Every subsequence of $U_n \longrightarrow +\infty$

6/ If $U_n \longrightarrow -\infty$ then Every subsequence of $U_n \longrightarrow -\infty$

Proof : admitted.

Remark See All indeterminate forms and some usual limits on the course matiral n° 1.

Exercise Study the nature of the geometric sequence $(U_n)_{n\in\mathbb{N}}$ defined by $U_n = a^n$. In fact, we want to:

Show that:

1/ If $|a|<1$ then $U_n \to 0$.

2/ If $a>1$ then $U_n \to +\infty$.

3/ If $a<-1$ then the limit of U_n doesn't exist.

4/ If $a=1$ then $U_n \to 1$.

5/ If $a=-1$ then the limit doesn't exist.

Solution

1/ suppose $|a|<1$. We get two cases:

- if $a=0$ then $U_n \longrightarrow 0$.
- if $a\neq 0$, let $\varepsilon>0$, we are looking for an $N\in\mathbb{N}$ such that $\forall n\geq N$, $|a^n|<\varepsilon$.

$$|a^n|<\varepsilon \iff |a|^n<\varepsilon$$
$$\iff n\ln|a| < \ln\varepsilon.$$
$$\iff n > \frac{\ln\varepsilon}{\ln|a|} \quad \text{because } \ln|a|<0$$

it is then sufficient to take $N = E\left(\frac{\ln\varepsilon}{\ln|a|}\right)+1$.

2/ If $a>1$, let $A>0$, we are looking for an $N\in\mathbb{N}$ such that: $\forall n\geq N \quad a^n>A$

$$a^n>A \iff n\ln a > \ln A$$
$$\iff n > \frac{\ln A}{\ln a} \quad (\text{because } \ln a>0)$$

So, take $N = E\left(\frac{\ln A}{\ln a}\right)+1$

3/ If $a<-1$, One has $U_n = a^n = \begin{cases} (-a)^{2k} & \text{if } n=2k \\ -(-a)^{2k+1} & \text{if } n=2k+1. \end{cases}$

$= \left[-(-a)\right]^n$

$= (-1)^n(-a)^n$

$a<-1 \iff (-a)>1$. According to 2/ One gets:

$(-a)^{2k} \longrightarrow +\infty$ and $-(-a)^{2k+1} \longrightarrow -\infty$

So $(U_n)_n$ divergence because there is no limite. It's a second type divergence.

4/ If $a=1$ then $\forall n \in \mathbb{N}$ $U_n = 1$ so $U_n \longrightarrow 1$.

5/ If $a=-1$ then $U_n = \begin{cases} -1 & \text{if } n = 2k+1 \\ 1 & \text{if } n = 2k. \end{cases}$

by the same argument as in 3/ (U_n) performs a second type divergence.

VII/ Adjacent Sequences

① **Definition**: Let be $(U_n)_n$ and $(V_n)_n$ two number sequences.

we say that $(U_n)_n$ and $(V_n)_n$ are adjacent iff:

1/ One of them is $\nearrow$ and the other one is $\searrow$.

2/ $\lim (U_n - V_n) = 0$

② **Theorem 11:**

> If $(U_n)_n$ and $(V_n)_n$ are adjacent sequences, then, they converge to the same limit $l \in \mathbb{R}$.

Proof suppose that $(U_n)_n$ is $\nearrow$ and $(V_n)_n$ is $\searrow$.

then: $\forall n \in \mathbb{N}: \begin{cases} U_0 \leq U_n \leq U_{n+1} \\ V_{n+1} \leq V_n \leq V_0 \end{cases}$; we will show that:

$\forall n \in \mathbb{N}, \ U_n \leq V_n$. Indeed, pose $(W_n)_{n\in\mathbb{N}}$ / $W_n = V_n - U_n$, $\forall n \in \mathbb{N}$

So $W_{n+1} = V_{n+1} - U_{n+1} \leq \underbrace{V_n - U_n}$

because: $\begin{cases} V_{n+1} \leq V_n \\ -U_{n+1} \leq -U_n \end{cases}$

$\Rightarrow W_{n+1} \leq W_n$

So $(W_n)_{n\in\mathbb{N}}$ is $\searrow$, And since $W_n \longrightarrow 0$ then $\forall n \in \mathbb{N}$ $W_n \geq 0$.

So, $\forall n \in \mathbb{N}$, $U_n \leq V_n$. We then get

$\forall n \in \mathbb{N}$ $U_0 \leq U_n \leq V_n \leq V_0$. This means that:

$\begin{cases} U_n \text{ is } \nearrow \text{ and upper bounded (by } V_0) \Rightarrow (U_n)_n \text{ converges} \\ V_n \text{ is } \searrow \text{ and lower bounded (by } U_0) \Rightarrow (V_n)_n \text{ converges.} \end{cases}$

In addition, $\lim (U_n - V_n) = \lim U_n - \lim V_n = 0$

$\boxed{\Rightarrow \lim U_n = \lim V_n}$

Recurring sequences

Definition Let be a map $f : D \subset \mathbb{R} \to \mathbb{R}$ such that:

$\boxed{f(D) \subset D}$. (We say that D is stable by f, or invariant by f)

a recurring sequence $(U_n)_{n \geq 0}$ is defined by:

1/ a given $U_0 \in D$

2/ a relation given by $\forall n \in \mathbb{N}$, $U_{n+1} = f(U_n)$.

Examples

① $\begin{cases} U_{n+1} = U_n^2 + 1 \\ U_0 = 1 \end{cases}$

The function associated with this sequence is $f(x) = x^2 + 1$ and $D = [1, +\infty[$. ← At home, verify that $f(D) \subset D$. here

② $\begin{cases} U_{n+1} = 1 + \dfrac{1}{U_n} \\ U_0 > 0 \end{cases}$

The function associated with this sequence is $f(x) = 1 + \dfrac{1}{x}$ and $D =]0, +\infty[$ ← and here.

Theorem

If $(U_n)_n$ is a recurring sequence (as in the definition above) Converges to $\ell \in D$ and f is continious on D then ℓ is a solution of the equation $f(x) = x$.

Note

1/ The solutions of the equation $f(x) = x$ are called fixed points of f.

2/ Let a be a fixed point of f.

if $U_0 \leq a$ then, $\forall n \in \mathbb{N}$ $U_n \leq a$ ↙ important

Theorem

Let $(U_n)_n$ be a recurring sequence, and f its associated map.

If f is ↗ then $(U_n)_n$ is monotone and:

1°/ when $U_0 \leq U_1$ then $(U_n)_n$ is ↗.

2°/ when $U_0 \geq U_1$ then $(U_n)_n$ is ↘

Proof (Simple) Suppose that $U_0 \leq U_1$.

We proove that $(U_n)_n$ is increasing by induction on n.

- $\rightarrow$ One has $U_0 \leq 1$
- $\rightarrow$ Suppose that $U_n \leq U_{n+1}$
- $\rightarrow$ Since $U_n \leq U_{n+1}$ then $f(U_n) \leq f(U_{n+1})$ (because f is $\nearrow$)

  $\Rightarrow U_{n+1} \leq U_{n+2}$
- $\rightarrow$ We conclude, by induction that $\forall n \in \mathbb{N}$ $U_n \leq U_{n+1}$

Same thing, if $U_0 \geq U_1$.

Theorem

Let $(U_n)_{n\in\mathbb{N}}$ be a recurring sequence and f its associated map.

If f is $\searrow$ Then $(U_n)_n$ is not monotone.

Furthermore $(U_{2n})_n$ and $(U_{2n+1})_n$ are monotonous with Contrary variations. That is to say that:

- If $U_0 \leq U_2$ then $(U_{2n})_n$ is $\nearrow$ and $(U_{2n+1})_n$ is $\searrow$
- If $U_0 \geq U_2$ then $(U_{2n})_n$ is $\searrow$ and $(U_{2n+1})_n$ is $\nearrow$

Proof If $U_0 \leq U_1 \Rightarrow f(U_0) \geq f(U_1)$ (because f is $\searrow$)

$$\Rightarrow U_1 \geq U_2$$

But when $U_1 \geq U_2 \Rightarrow f(U_1) \leq f(U_2)$

$$\Rightarrow U_2 \leq U_3 \quad \ldots \text{ and so on}$$

So $(U_n)_n$ is not monotone.

Now, Put $\begin{cases} v_n = U_{2n} & \forall n \in \mathbb{N} \\ w_n = U_{2n+1} & \forall n \in \mathbb{N} \end{cases}$

One has:

$$v_{n+1} = U_{2(n+1)} = U_{2n+2} = f(U_{2n+1}) = f(f(U_{2n}))$$
$$= (f \circ f)(U_{2n})$$
$$= (f \circ f)(v_n)$$

$$\Rightarrow \boxed{v_{n+1} = (f \circ f)(v_n)}$$

$$w_{n+1} = U_{2(n+1)+1} = U_{2n+3} = f(U_{2n+2}) = f(f(U_{2n+1}))$$
$$= (f \circ f)(U_{2n+1})$$
$$= (f \circ f)(w_n)$$

$$\Rightarrow \boxed{W_{n+1} = (f\circ f)(W_n)}$$

This means that $(V_n)_n$ and $(W_n)_n$ are recurring sequences with the associated map $(f\circ f)$.

And since f is $\searrow$ then $(f\circ f)$ is $\nearrow$.

So, if $U_0 \leq U_2 \Rightarrow \begin{cases} f(U_0) \geq f(U_2) \\ (U_{2n})_n \nearrow \text{ (by induction as before)} \end{cases}$

$\Rightarrow \begin{cases} U_1 \geq U_3 \\ (U_{2n})_n \nearrow \end{cases}$

as $U_1 \geq U_3$ by induction, we show easily that $(U_{2n+1})_n$ is $\searrow$

Same thing when $U_0 \geq U_2$.

Examples

① $\begin{cases} U_{n+1} = U_n^2 + 1 \\ U_0 = 1 \end{cases}$

Could be prooved by induction!

We can see clearly that, $\forall n \in \mathbb{N}$ $U_n \geq 1$, so, consider

$f(x) = x^2 + 1, \quad x \in D = [1, +\infty[$.

$\Rightarrow f'(x) = 2x$

x	1		$+\infty$
$f'(x)$	2	$+$	
$f(x)$	2	$\nearrow$	$+\infty$

$f(D) = [2, +\infty[\subset D$!

f is $\nearrow$ on $D \Rightarrow (U_n)_n$ is monotone

and since $U_1 = 2 > U_0$ so $(U_n)_n$ is $\nearrow$.

f is continious. So if $(U_n)_n$ converges, it must converge to a fixed point of f. Let us solve the equation $f(x) = x$.

$f(x) = x \Leftrightarrow x^2 - x + 1 = 0$; $\Delta = -3 < 0$. There is no real solutions. This means that f doesn't have fixed points.

We may conclude that $(U_n)_n$ diverges to $+\infty$.

Exercices at home and at the amphi:

Discuss the nature of the following recurring sequences: ① $\begin{cases} U_{n+1} = \sqrt{2U_n + 3} \\ U_0 \in \mathbb{R}^+ \end{cases}$; ② $\begin{cases} U_{n+1} = 1 + \dfrac{2}{U_n} \\ U_0 = 1 \end{cases}$

Correction

① $\begin{cases} U_{n+1} = \sqrt{2U_n+3} \\ U_0 \in \mathbb{R}^+ \end{cases}$ | ② $\begin{cases} U_{n+1} = 1 + \dfrac{2}{U_n} \\ U_0 = 1 \end{cases}$

Consider $f(x) = \sqrt{2x+3}$ On $D = [0, +\infty[$.

$2x+3$: $-$ $\,-\frac{3}{2}\,$ $+$, 0

- f is continious on D.
- $f'(x) = \frac{1}{2}(2)\dfrac{1}{\sqrt{2x+3}} = \dfrac{1}{\sqrt{2x+3}}$
- $f(D) = [\sqrt{3}, +\infty[\subset D$

x	0	$+\infty$
$f'(x)$	$\frac{1}{\sqrt{3}}$	$+$
$f(x)$	$\sqrt{3}$ ↗	$+\infty$

Since f is increasing, then $(U_n)_n$ is monotone.
Since we do not know U_0 nor U_1, to find the monotony of U_n, we must find the sign of $(U_1 - U_0)$ or equivalently the sign of $(U_{n+1} - U_n)$ (Since $(U_n)_n$ is monotone). So let us check:

$$f(x) - x = \sqrt{2x+3} - x$$

$$f(x) - x = 0 \iff \sqrt{2x+3} - x = 0 \iff \underbrace{\sqrt{2x+3}}_{>0} = \underbrace{x}_{>0}$$

$$\iff 2x + 3 - x^2 = 0$$

$$\Delta = 4 - 4(-1)(+3) = 16$$

$$x_1 = \frac{-2+4}{(-2)} < 0, \quad x_2 = \frac{-2-4}{(-2)} = 3 \in D$$

$x_1 \notin D \Rightarrow$ refused; $x_2 \Rightarrow$ accepted.

$f(x) - x$: 0 $\;+\;$ 3 $\;-\;$

- So, if $U_0 \in [0, 3]$, $(U_n)_n$ is increasing and bounded by 3 (because 3 is a fixed point of f, and $U_0 \leq 3$).
  This means that $(U_n)_n$ is convergent to 3.

- If $U_0 \in]3, +\infty[$, $(U_n)_n$ is decreasing.
  but, $\forall n \in \mathbb{N}$, $U_n \geq U_0 > 3$ because f is ↗.
  This means that $(U_n)_n$ is convergent, and since 3 is the only fixed point in D, then $(U_n)_n \longrightarrow 3$.

Conclusion $(U_n)_n$ is a convergent sequence, that converges to 3.

② $\begin{cases} U_{n+1} = 1 + \dfrac{2}{U_n} \\ U_0 = 1 \end{cases}$

Consider $f(x) = 1 + \dfrac{2}{x}$ On $D = [1, +\infty[$.

- f is continious on D.
- $f'(x) = -\dfrac{2}{x^2}$

x	1		$+\infty$
f'	-2	$-$	
f	3	$\searrow$	1

- $f(D) =]1, 3] \subset D$
- f is decreasing so $(U_n)_n$ is not monotone. But $(U_{2n})_n$ and $(U_{2n+1})_n$ are monotone with contrary variations.
- One has, $U_0 = 1 \Rightarrow U_1 = 3 \Rightarrow U_2 = \frac{5}{3}$ So, Since $U_0 \leq U_2$ then $(U_{2n})_n$ is $\nearrow$, which implies that $(U_{2n+1})_n$ is $\searrow$
- It remains to find the fixed points of $(f \circ f)$:

$$(f \circ f)(x) = x \iff 1 + \frac{2}{f(x)} = x$$

$$\iff \frac{3x+2}{x+2} = x \quad \text{(multiplying by } x\text{, since } x\text{)}$$

$$\iff x^2 - x - 2 = 0 \qquad \Delta = 1 - 4(1)(-2) = 9$$

$$\iff x_1 = \frac{1-3}{2} = -1 \notin D, \quad x_2 = \frac{1+3}{2} = 2 \in D$$

There is only one fixed point of $(f \circ f)$ that belongs to D, which is

- Since $U_0 = 1 \leq 2$ and 2 is fixed point of $(f \circ f) \Rightarrow U_{2n} \leq 2$, So $(U_{2n})_n$ is increasing and upper bounded by 2. this means that it converges to the only fixed point of $(f \circ f)$ i.e. $(U_{2n})_n \to 2$.

Now, it is easy to see that $\forall n \in \mathbb{N}$ $U_n \geq 1$ and since (U_{2n+1}) is decreasing (and lower bounded by 1) then it converges to the only fixed point of $(f \circ f)$ which is 2. i.e. $(U_{2n+1})_n \to 2$

- Since $(U_{2n}) \to 2$ and $(U_{2n+1}) \to 2$, then $(U_n)_{n \in \mathbb{N}} \to 2$.

Exercise Study $\begin{cases} U_{n+1} = 1 + \dfrac{U}{n} \\ U_0 > 0 \end{cases}$