Abou Bekr Belkaid University - Tlemcen | Academic year: 2024 - 2025
Faculty of Sciences | Module: Algebra 1
Department of Mathematics | L1 Math

Tutorial sheet N°1 : Logic and reasoning

Exercise 1

(1) Let P, Q and R be propositions. Show that

$$\left[\left(P \wedge \overline{Q}\right) \Rightarrow R\right] \Leftrightarrow \left[\overline{P} \vee \left(\overline{Q} \Rightarrow R\right)\right]$$

using :
(i) The truth table.
(ii) The definition of an implication, its negation, De Morgane's laws and the properties of logical connectors.
(2) Using the truth table, show that the following implication is always true :

$$\left[(P \Rightarrow Q) \wedge \left(\overline{P} \Rightarrow Q\right)\right] \Rightarrow Q$$

Application : Let $n \in \mathbb{N}^*$, show that :

(i) $n(n+1)$ is even

Exercise 2 $\left(\textit{Connectors NAND } \left(\textit{NOT AND}\right) \textit{ and NOR } \left(\textit{NOT OR}\right)\right)$

For two propositions P and Q, we define the connectors $NAND$ $\left(NOT\ AND\right)$ and NOR $\left(NOT\ OR\right)$ by

$$P\ NAND\ Q \Leftrightarrow P \uparrow Q \Leftrightarrow \overline{P \wedge Q} \text{ and } P\ NOR\ Q \Leftrightarrow P \downarrow Q \Leftrightarrow \overline{P \vee Q}$$

(1) Draw up the truth tables of the two connectors $NAND$ and NOR.
(2) Determine $P \uparrow P, P \downarrow P, \overline{P \uparrow Q}$ and $\overline{P \downarrow Q}$
(3) Express $\overline{P}, P \vee Q, P \wedge Q$ and $P \Rightarrow Q$ using connectors $NAND \uparrow$ et $NOR \downarrow$.
Remark : The connectors $NAND \uparrow$ and $NOR \downarrow$ are important in computing and electronics.

Exercise 3

(I) Give the negation and the truth value of the following propositions (predicates):

$$P_1 : \exists\, x \in \mathbb{R}, x^2 - 2 = 0 \qquad P_2\ \forall x \in \mathbb{R}^+, \exists n \in \mathbb{N} : n > x$$

$$Q_1 : \forall\, n \in \mathbb{N}, \frac{n+1}{2} \in \mathbb{N} \qquad Q_2 : \exists\, x \in [0, \pi], \cos x + \sin x = 1$$

$$R_1 : \forall\, x \in \mathbb{R}, \exists\, y \in \mathbb{R} : y^2 = x \qquad R_2 : \exists\, x \in \mathbb{R}, \forall\, y \in \mathbb{R} : y^2 = x$$

(II) Using quantifiers and mathematical symbols, write the following expressions:
(1) For any real number x, there exists a naturel number n such that, x is less than n.
(2) For all real number x, there exists a relative integer n, such that x is greater than or equal to n and less than n plus 1.
(3) Between two real numbers, there exists at least one rational number.
(4) Some real numbers are greater than or equal to their square.
(5) $\left(\textbf{supp}\right)$The cube of any real number is positive.
(6) $\left(\textbf{supp}\right)$There exists a unique natural number less than all others natural numbers.
(7) $\left(\textbf{supp}\right)$There exists a non-zero natural number that divides every non-zero relative integer.
(8) $\left(\textbf{supp}\right)$The equation $x^2 - 2 = 0$ admits at least one rational solution.

Exercise 4

(I) $\left(Direct\ reasoning\right)$ Prove the following assertions :

$$(1)\ \forall x \in \mathbb{R}, |x-1| \leq x^2 - x + 1$$
$$(3) \forall x \in \mathbb{R}^+, \frac{1}{1+\sqrt{x}} = 1 - \sqrt{x} \Rightarrow x = 0$$

(II) $(Contra$ positive $and\ contradiction)$

(1) Show that $\forall x \in \mathbb{R}, \left(x \neq -5 \text{ and } x \neq -8\right) \Rightarrow \dfrac{x+2}{x+5} \neq 2.$

(2) Show that: $\forall x, y \in \mathbb{R}, x \neq y \Rightarrow (x+1)(y-1) \neq (x-1)(y+1).$

Exercise 5

For $n \in \mathbb{N}^*$. Consider the statement : If $n^2 - 1$ is not a multiple of 4, then n is even.

(1) Write this statement in the form of an implication $P \Rightarrow Q$.

(2) Write the contrapositive and converse of this implication $P \Rightarrow Q$.

(3) Show the implication $P \Rightarrow Q$ by contraposition.

(4) Is the converse of the implication $P \Rightarrow Q$ true?

Exercise 6

For $n \in \mathbb{N}$, we consider the proposition

$$A :\ n^3 \text{ is even} \Rightarrow\ n \text{ is even.}$$

(1) Write the negation of the proposition A.

(2) Write the contrapositive of the proposition A.

(3) Show that proposition A is true.

(4) Show by contradiction that $\sqrt[3]{2}$ is irrational number.

Exercise 7 $\left(\text{Proof by induction}\right)$

(1) Let $x > 0$. Show by induction that : $\forall\ n \in \mathbb{N}^*,\ (1+x)^n \geq 1 + n.x$ $\left(\text{Bernoulli inequality}\right)$

(2) $\forall n \in \mathbb{N}^*$, we consider the sum $S_n = \sum_{k=1}^{n} \dfrac{1}{k(k+1)} = \dfrac{1}{1.2} + \dfrac{1}{2.3} + \dfrac{1}{3.4} . + \ldots + \dfrac{1}{n.(n+1)}$

(i) Calculate $S_1,\ S_2,\ S_3$ and S_4.

(ii) Propose or conjecture a formula in n for S_n. (iii) Prove this formula by induction.

(3) $\left(\textbf{supp}\right)$ Let $x_1,\ x_2, \ldots x_n \in \left[0,\ 1\right]$.

Show by induction that : $\forall\ n \geq 1,\ \prod_{k=1}^{n}(1 - x_k) \geq 1 - \sum_{k=1}^{n} x_k.$

Indication :

$$\prod_{k=1}^{n+1}(1 - x_k) = \prod_{k=1}^{n}(1 - x_k) + (1 - x_{n+1}) \sum_{k=1}^{n} x_k + x_{n+1} = \sum_{k=1}^{n+1} x_k$$

Exercise 8 $\left(\textbf{different types of reasoning for an obvious statement}\right)$

Suggest the following statement various demonstrations : $P(n)$: $\forall n \in \mathbb{N}$, the integer $5^n + 1$ is even.

(i) Proof by induction (ii) Proof by contradiction (iii) Using a remarkable identity to factor $5^n - 1$

(iv) Using a method distinct from the previous three.

Supplementary exercises

Exercise 1

(1) Using the truth table, which of the following three propositions are equivalent?
$[(P \vee Q) \Rightarrow R]$, $[(P \Rightarrow R) \vee (Q \Rightarrow R)]$ and $[(P \Rightarrow R) \wedge (Q \Rightarrow R)]$.
(2) (i) Using the truth table, show that the following implication is always true :

$$[(P \Rightarrow Q) \wedge (R \Rightarrow Q)] \Rightarrow [(P \vee R) \Rightarrow Q]$$

(ii) Deduce that

$$\left[(P \Rightarrow Q) \wedge \left(\overline{P} \Rightarrow Q\right)\right] \Rightarrow Q$$

(iii) Application: Let $n \in \mathbb{N}^*$, show that:

$$\frac{n(n+1)(n+2)}{3} \in \mathbb{N}$$

(3) Show that the following implications : (A) $(P \wedge Q) \Rightarrow \overline{Q}$ and (B) $\left(P \wedge \overline{Q}\right) \Rightarrow Q$ are both true if and only if P is false.
(4) Is the following proposition true? $(P \wedge Q) \Rightarrow (\overline{P} \vee Q)$
(5) Are the following propositions tautologies?

$$[(P \Rightarrow Q) \wedge P] \Rightarrow Q \quad \text{and} \quad (P \vee \overline{P}) \vee Q \Rightarrow (P \wedge \overline{P}) \wedge \overline{Q}$$

Exercise 2 (*The exclusive OR* (*XOR*) *denoted* $\oplus$)

For two propositions A and B, we define the exclusive OR (XOR) denoted $\oplus$:by $A \oplus B$ is true if A is true or B is true and not both true at the same time.
- Show that :

$$A \oplus B \Leftrightarrow (A \wedge \overline{B}) \vee \left(\overline{A} \wedge B\right) \Leftrightarrow (A \vee B) \wedge \left(\overline{A} \vee \overline{B}\right)$$

$$A \oplus B \Leftrightarrow B \oplus A \qquad A \oplus F \Leftrightarrow A \qquad A \oplus A \Leftrightarrow F$$

where A,B,C and F are propositions with F is false.

Exercise 3

Show that :
(1) $\forall a,b \in \mathbb{R}, a^2 + b^2 = 1 \Rightarrow |a+b| \leq \sqrt{2}$
(2) $\forall x \in \mathbb{R},\ x + \frac{1}{x} \geq 2.$ .....(3) $\forall x,y \in \mathbb{R}, \sqrt{x^2+1} + \sqrt{y^2+1} = 2 \Rightarrow x = y = 0$
(4) Solve the following equation in $\mathbb{R}$: $\sqrt{x^2+1} = 2x.$

Exercise 4

Show that the following propositions are false.
(i) $\forall x \in [0,1],\ x^2 \geq x$ (ii) $\forall x,y \in \mathbb{R}, x^2 + y^2 \geq x + y$ (iii) $\forall a,b \in \mathbb{R}, \sqrt{a^2+b^2} = a + b$
(iv) The function f defined on $\mathbb{R}$ by : $f(x) = x^2 + 2x,$ is neither even nor odd.
(v) $\forall\ n \in \mathbb{N}$, the integer $n^2 + n + 11$ is prime.

Exercise 5

Show by induction that
(1) $\forall\ n \in \mathbb{N},\ n^2 < 3^n.$ (Indication : $\forall\ n \geq 2,\ n^2 \geq 2n$ and $n^2 > 1$).
(2) $\forall\ n \in \mathbb{N}^*,\ \sum_{k=1}^{n} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} .+\ldots+ \frac{1}{n^2} \leq 2 - \frac{1}{n}.$

Exercise 6

Consider the following propositions:

$$P : \forall x \in \mathbb{R}^{+}, \sqrt{x^2+1} + x > 0,$$
$$Q : \forall x \in \mathbb{R}^{-*}, \sqrt{x^2+1} + x > 0 \text{ and}$$
$$R : \forall x \in \mathbb{R}, \sqrt{x^2+1} + x > 0$$

(1) Show by direct reasoning that P is true.
(2) Write the negation of the proposition Q.
(3)Show by the contradiction that Q is true.
(4) Deduce that R is true.

Exercise 7

(I) (1) Recall the sum $\sum_{k=1}^{n} k$.
(2) Verify : $2n^2 + 7n + 6 = (n+2)(2n+3)$.
(3) Show by induction that $\forall n \in \mathbb{N}^*$:

$$S_n = \sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

(4) Deduce the sum $\sum_{k=1}^{n} k(k+1)$.
(II) Show by induction that : $\forall n \in \mathbb{N}$, $4^n + 6n - 1$ is divisible by 9.

Exercice 8

For $n \in \mathbb{N}^*$, we consider the sum

$$S_n = \sum_{k=1}^{n} k(k+1) = 1.2 + 2.3 + 3.4 + \ldots + n(n+1).$$

(1) Calculate S_1, S_2 et S_3 .
(2) Prove by induction that $\forall n \in \mathbb{N}^*$:

$$S_n = \frac{n(n+1)(n+2)}{3}.$$

(3) Deduce the sum $\sum_{k=1}^{n} k^2$.
$\left(\text{Indication : } \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \text{ and } \sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k\right).$

Exercise 9

Let $n \in \mathbb{N}^*$.
(1) By a direct proof, show that :

$$n \text{ is a multiple of } 3 \Rightarrow n^3 \text{ is a multiple of } 3.$$

(2) Using the contrapositive, show that :

$$n^3 \text{ is a multiple of } 3 \Rightarrow n \text{ is a multiple of } 3.$$

(3) Show by contadiction that $\sqrt[3]{3}$ is a irrational number .