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Module: Algebra 1
L1 Math+MI

Tutorial sheet N°2 : Sets and maps

Exercise 1

(1) Write the following sets in extension:

$A = \left\{ n \in \mathbb{Z} \setminus -\frac{3}{\sqrt{2}} < n - 1 < 2\sqrt{\pi} \right\}$ and $B = \left\{ z \in \mathbb{C} : z^4 - 1 = 0 \right\}$.

(2) Write the following sets in comprehension:

$C = \left\{ \frac{4}{5}, \frac{6}{7}, \frac{8}{9}, \frac{10}{11}, \frac{12}{13}, \frac{14}{15} \right\}$ and $D = \left\{ -1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, -\frac{1}{7}, \dots \right\}$.

(3) Let $E = \{1,2,3\} \subset \mathbb{N}$. Find $P(E)$ the set of power of E.

(4) Let A, B and C be parts of $P(E)$ given by :

$A = \left\{ X \in P(E) : 1 \in X \right\}$, $B = \left\{ X \in P(E) : 2 \notin X \right\}$, $C = \left\{ X \in P(E) : 1 \notin X \text{ and } 3 \in X \right\}$.

- Give the sets A, B and C in extension.

(5) Find

$A \cap B, .A \setminus B, B \cup C$, and $\overline{C} = C_{P(E)}(C)$, the complement of C relative to $P(E)$.

Exercise 2

For $n \in \mathbb{N}^*$, let's pose $A_n = \left] \frac{1}{n+1}, \frac{1}{n} \right]$.

(i) Show that the family of sets $\left\{ A_n, n \in \mathbb{N}^* \right\}$ is a recovery of the interval $]0, 1]$.

(ii) Deduce that $\left\{ A_n, n \in \mathbb{N}^* \right\}$ is a partition of $]0, 1]$.

Exercise 3

Let $A, B,$ and C be three parts of a set E. Show that :

(1) $\left[A \cap B = A \cap C \text{ and } A \cup B = A \cup C \right] \Rightarrow B = C$

(2) $A \cap B = A \cup B \Rightarrow A = B$, using :

(i) Direct proof.

(ii) The contrapositive.

(3) $(Supp)$ $A \cup B = A \cap C \Leftrightarrow B \subset A \subset C$.

Exercise 4 $(Homework)$

We consider the two sets E and F defined by :

$$E = \left\{ x \in \mathbb{R} : \left| 1 - \frac{x}{2} \right| \leq 1 \right\} \text{ and } F = \left\{ x \in \mathbb{R} : \exists t \in \mathbb{R}^+, x = t + 2 \right\}.$$

(1) Write E as an interval $[a, b]$.

(2) Show that $F = [2, +\infty[$.

(3) Find $E \cap F$, $E \cup F$, $E \setminus F$, $F \setminus E$, $F \triangle E$ et $C_{\mathbb{R}^+}(E)$, the complement of E relative to $\mathbb{R}^+$.

Exercise 5

Let D_{24} denote the set of divisors of 24. Consider the two sets E and F defined by :
$E = \{n \in D_{24} : n \text{ is even}\}$ and $F = \{n \in \mathbb{N} \setminus n \text{ is prime and } n < 9\} \cup \{1, 10\}$.
Let $f : E \to F$ be a map defined by its graph $G = \{(2,3),(4,5),(6,1),(8,5),(12,7),(24,10)\}$.
(1) Check that f is indeed a map.
(2) Is f injective (one-to-one) ? surjective (onto)?
(3) Determine $f(6), f(\{6\}), f(\{n \in E \setminus n \text{ divides } 8\})$ and $f(E)$.
(4) Determine $f^{-1}(1), f^{-1}(\{1\}), f^{-1}(\{5\}), f^{-1}(\{n \in \mathbb{N} \setminus n \text{ is prime and } n < 9\})$ and $f^{-1}(F)$.

Exercise 6

Let's note by $J =]1, +\infty[$. Let f and $g : J \to J$ be two maps defined by :

$$\forall x \in J,\ f(x) = 1 + \frac{2}{\sqrt{x} - 1} \text{ and } g(x) = \left(\frac{\sqrt{x} + 1}{\sqrt{x} - 1}\right)^2.$$

(1) Determine $f([2,4[)$ and $g^{-1}(\{9\})$.
(2) Show that f is a bijection from J into J and determine its inverse map.
(3) Check that : $\forall x \in J,\ g(x) = (f(x))^2$.
(4) Deduce that g is a bijection from J into J and determine its inverse map.

Exercise 7

Let's note by $U =]0, +\infty[\times]0, +\infty[$ and Let $f : U \to U$ be a map defined by :

$$f(x, y) = \left(xy, \frac{y}{x}\right)$$

(1) Show that f is injective (one-to-one)? $\left(\text{We can pose } t = \frac{y}{x}\right)$.
(2) Is f surjective (onto)? If yes, determine f^{-1}.

Exercise 8 (*Homework*)

Let f and g be two maps defined by :

$$f : \mathbb{R}\setminus\{3\} \to \mathbb{R}/\{4\} \quad x \mapsto \frac{4x+1}{x-3} \qquad \text{and} \qquad g : \mathbb{R}/\{4\} \to \mathbb{R}\setminus\{3\} \quad x \mapsto \frac{3x+1}{x-4}$$

(1) Show that $\forall y \in \mathbb{R}\setminus\{4\}, \dfrac{3y+1}{y-4} \neq 3$.
(2) Find for $x \in \mathbb{R}\setminus\{1, 3\}$ the image of $3x$ by f and for $x \in \mathbb{R}\setminus\{-2, 2, 4\}$ the image of x^2 by g.
(3) Determine the preimages $y = 1$ by f and by g.
(4) Determine $f \circ g$ and $g \circ f$.
(5) Show that f is injective (one-to-one).
(6) Is f surjective (onto)?
(7) Is f bijective? If yes, determine f^{-1}.

SUPPLEMENTARY EXERCISES

Exercise 1

Let $A = \left\{(x,y) \in \mathbb{R}^2 \backslash\ x^2 + y^2 \leq 1\right\}$ be a set.

(1) Do the pairs $(1,0)$ and $(0,1)$ belong to A?

(2) Show that A can not be the cartesian product of two parts of $\mathbb{R}$.

Indication : By contradiction and notice that $(1,1) \notin A$.

Exercise 2

Consider the following two sets :

$$A = \left\{x \in \mathbb{R} \backslash \exists\, t \in \left[0,\ 1\right] : x = t + 2\right\} \text{ and}$$
$$B = \left\{x \in \mathbb{R} \backslash \left|x - \frac{5}{2}\right| \leq \frac{1}{2}\right\}.$$

(1) Write the set B as an interval $\left[a,\ b\right]$.

(2) Show that $A = B$.

Exercise 3

Let $A, B,$ and C be three parts of a set E. Show that :

(1) $(A \backslash B) \backslash C = A \backslash (B \cup C)$

(2) $C_E(A \cap B) = C_E(A) \cup C_E(B)$

(3) $A \triangle B = A \cap B \Leftrightarrow A = B = \varnothing$

Exercise 4

(1) Consider the following sets:

$A = \left\{\dfrac{5n+8}{8n-1}, n \in \mathbb{N}\right\}$ and $B = \left\{\dfrac{2n+4}{2n-1}, n \in \mathbb{N}\right\}$.

(i) Does $\dfrac{17}{3} \in A$? $\dfrac{18}{15} \in A$? $\dfrac{43}{25} \in B$? $\dfrac{42}{37} \in B$?

(ii) Show that $\frac{6}{5}$ is a common element between sets A and B.

(2) Let $C = \left\{\frac{\pi}{4} + \frac{2k\pi}{5}, k \in \mathbb{Z}\right\}$ and $D = \left\{\frac{\pi}{2} + \frac{2k\pi}{5}, k \in \mathbb{Z}\right\}$, be two sets.

- Show that $A \cap B = \varnothing$.

Exercise 5

Consider the two parts of $\mathbb{R}^2$, $E = [0,1]$ and $F = [0,2]$

(1) Draw $E \times F$ and $E \times E$.

(2) Let $f : E \longrightarrow F$ and $g : F \longrightarrow E$ be two maps.

$x \longmapsto 2 - x \qquad x \longmapsto (x-1)^2$

(3) Specify $g \circ f$ and $f \circ g$. Do we have $g \circ f = f \circ g$ and $g \circ f = g$?

(4) Determine $f^{-1}(\{0\})$ and $g^{-1}\left(\left]0, \frac{1}{2}\right[\right)$.

(5) Show that : $g \circ f$ is bijective and specify $(g \circ f)^{-1}$.

Exercise 6

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a map defined by : $f(x) = \dfrac{2x}{1+x^2}$.

(1) Determine $f(2)$ and $f\left(\frac{1}{2}\right)$. f is it injective (one-to-one)?

(2) Solve in $\mathbb{R}$: $f(x) = 2$. f is it surjective (onto)? Show that $f(\mathbb{R}) = [-1,1]$.

(3) Show that the application g defined on $[-1,1]$ in $[-1,1]$ by: $f(x) = g(x)$ is bijective and determine its inverse g^{-1}.

Exercise 7

Let $f : \mathbb{R} \to \mathbb{R}^+$ be a map defined by :

$$\forall x \in \mathbb{R},\ f(x) = \frac{1}{1 + \sqrt{4 + x^2}}.$$

(1) Determine $f\left(\{x \in \mathbb{R} \setminus x^2 = 4\}\right)$, $f(\mathbb{R}^+)$ and $f^{-1}\left(\{y \in \mathbb{R}^+ \setminus |y| = 1\}\right)$.

(2) Is f injective? surjective? bijective?

(3) Let $g = f_{\setminus \mathbb{R}^+} : \mathbb{R}^+ \to J$ where $J = f(\mathbb{R}^+)$, the restriction of f on $\mathbb{R}^+$.

- Show that g is bijective and determine g^{-1}.

Exercise 8

We define the maps $f : \mathbb{R} \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to [2, +\infty[$ by :

$$\forall x \in \mathbb{R},\ f(x) = 1 + \sqrt{1 + x + x^2}$$

$$\forall x \in \mathbb{R}^+,\ g(x) = 1 + \sqrt{1 + x^2}$$

(1) Find $f\left(\{x \in \mathbb{R} \setminus x^2 + x - \alpha(\alpha + 1) = 0, \text{ where } \alpha \in \mathbb{R}^{+*}\}\right)$ and $f^{-1}\left(\{y \in \mathbb{R}^+ \setminus 2|y| = 1\}\right)$.

(2) Is f injective? surjective? bijective?

(3) Show that g is bijective and determine g^{-1}.

Exercice 9

Let $h : \mathbb{R}^+ \to \left[\frac{1}{4}, +\infty\right[$ be a map defined by : $\forall x \in \mathbb{R}^+,\ h(x) = x + \sqrt{x} + \frac{1}{4}$.

(1) Check that : $\forall x \in \mathbb{R}^+,\ h(x) = \left(\sqrt{x} + \frac{1}{2}\right)^2$.

(2) Write the application h as the composite of two maps f and g : $h = g \circ f$.

(3) Show that f is a bijection and determine its inverse map.

(4) Show that g is a bijection and determine its inverse map.

(5) Deduce that h is a bijection of $\mathbb{R}^+$ in $\left[\frac{1}{4}, +\infty\right[$, and determine its inverse map.

Exercise 10

Let $f : \mathbb{R}^* \to \mathbb{R}$ and $g : \mathbb{R}^{*+} \to \mathbb{R} \setminus \{1\}$ be two maps.

$x \mapsto 1 - \frac{1}{x^2}$ $\qquad$ $x \mapsto 1 - \frac{1}{x^2}$

(1) Determine $f(\{-1, 1\})$ and $f^{-1}(\{1\})$.

(2) Is f injective? Surjective?

(3) Show that the map g is bijective and determine g^{-1}.

Exercise 11

(I) Let $f : \mathbb{R} \to \mathbb{R}$ be a map defined by : $f(x) = \dfrac{1}{\sqrt{1 + x^2}}$.

(1) Determine $f\left(\{x \in \mathbb{R} \setminus |x| = 1\}\right)$ and $f^{-1}\left(\{y \in \mathbb{R} \setminus y^3 = 8\}\right)$.

(2) Is f injective (one-to-one)? surjective (onto)? bijective?

(3) Determine $f\left(\left[1, \sqrt{3}\,\right]\right)$, $f\left(\left]-\sqrt{3}, -1\right]\right)$, $f\left(\left[-1, 2\sqrt{2}\,\right[\right)$, $f(\mathbb{R}^+)$, $f^{-1}(]0, 1])$ and $f^{-1}\left(\left[\frac{1}{2}, 1\right]\right)$.

(II) Let $g = f_{\setminus \mathbb{R}^+} : \mathbb{R}^+ \to J$ where $J = f(\mathbb{R}^+)$. (g is the restriction of f on $\mathbb{R}^+$).

(1) Show that g is bijective and determine g^{-1}.

(2) Determine $g^{-1}\left(\frac{1}{2}\right)$ by two methods.

(3) Calculate $g \circ g^{-1}(y)$ for $y \in J$ and $g^{-1} \circ g(x)$ for $x \in \mathbb{R}^+$.