Abou Bekr Belkaid University - Tlemcen
Faculty of Sciences
Department of Mathematics

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Module: Algebra 1
L1 Math+MI

Tutorial sheet N°3 : Binary relations

Exercise 1

We define in $\mathbb{R}^*$ the relation R by :

$$\forall x,\ y \in \mathbb{R}^*,\ x\, R\, y \iff x^2 - \frac{1}{x^2} = y^2 - \frac{1}{y^2}$$

(1)Show that R is an equivalence relation.
(2) Determine the equivalence class of $a \in \mathbb{R}^*$.

Exercise 2

In $\mathbb{R}$ we define the relations $\Re_1$ and $\Re_2$ by :

$$\forall x,\ y \in \mathbb{R},\ x\, \Re_1\, y \iff x - y \in \mathbb{N}$$
$$\forall x,\ y \in \mathbb{R},\ x\, \Re_2\, y \iff x - y \in \mathbb{Z}$$

(1) Show that $\Re_1$ is an order relation.
(2) Is this order total?
(3) Show that $\Re_2$ is not an order relation, but it is an equivalence relation.
(4) Determine the equivalence classes of 0, 1 and $a \in \mathbb{Z} \subset \mathbb{R}$.

Exercise 3

Let $\Re$ be the relation defined on $\mathbb{N}^*$ by :

$$\forall a,\ b \in \mathbb{N}^*,\ a\, \Re\, b \Leftrightarrow \exists\, n \in \mathbb{N} \text{ such : } a^n = b.$$

(1) Show that $\Re$ is an order relation on $\mathbb{N}^*$.
(2) Is this order total?
(3) Let the set $A = \{1,\ 4,\ 8\}$. Determine if they exist, $\max(A)$ and $\min(A)$ in the sense of this order.

Supplementary exercises

Exercise 1

(1) Say if the following relations are reflexive, symmetric, antisymmetric, transitive :
(i) $E = \mathbb{Z}$ and $x\Re y \iff x = -y$
(ii) $E = \mathbb{R}$ and $x\Re y \iff \cos^2 x + \sin^2 y = 1$
(iii) $E = \mathbb{N}$ and $x\Re y \iff \exists p, q \in \mathbb{N}^*, y = px^q$.
(2) Which of the above examples are order relations and equivalence relations?

Exercise 2

(1) Let $\Re$ be a reflexive binary relation defined on a set E such that :

$$\forall x,y,z \in E : x\Re y \text{ and } y\Re x \Rightarrow z\Re x.$$

Such a relation is called circular.

- Verify that $\Re$ is an equivalence relation.

(2) Let S be a reflexive and transitive relation defined in a set E and $\triangle$ another relation defined by :

$$\forall x,y \in E,\ x\triangle y \Leftrightarrow xSy \ \wedge ySx.$$

- Check that $\triangle$ is an equivalence relation.

Exercise 3

We define in $\mathbb{R}^*$ the relation $\Re$ by :

$$\forall x,y \in \mathbb{R}^*,\ x\Re y \Longleftrightarrow \frac{x}{y} > 0.$$

(1) Show that $\Re$ is an equivalence relation.

(2) Determine $cl(1)$, $cl(-2)$ and $cl(a)$ where $a \in \mathbb{R}^*$.? Deduce quotient set $\mathbb{R}^*/\Re$.

Exercise 4

In $\mathbb{R}$ we define the relation $\Re$ by :

$$\forall x,y \in \mathbb{R}, x\Re y \Leftrightarrow x \equiv y[2\pi] \Leftrightarrow \exists n \in \mathbb{Z} : x - y = 2\pi n.$$

(1) Show that $\Re$ is an equivalence relation.

(2) Determine $cl(0)$, $cl(\pi)$ et $cl(2\pi)$.

Exercise 5

In $\mathbb{R}^2$ we define the relation $\Re$ by :

$$\forall\ (x,y),\ (x',y') \in \mathbb{R}^2,\ (x,y)\Re(x',y') \Leftrightarrow x < x' \text{ or } \left(x = x' \text{ and } y \leq y'\right)$$

(1) Show that $\Re$ is an order relation.

(2) Is this order total?

Exercise 6

Let $\Re$ be the binary relation defined on $\mathbb{Z} \times \mathbb{N}^*$ by :

$$\forall (a,b),(a',b') \in \mathbb{Z} \times \mathbb{N}^*, (a,b)\Re(a',b') \Leftrightarrow ab' = a'b.$$

(1) Show that $\Re$ is an equivalence relation.

(2) Determine $cl(1,2)$ and $cl(-1,2)$.

Exercise 7

We define in $\mathbb{R}^2$the relation $\leq$ by :

$$\forall (x,y),(x',y') \in \mathbb{R}^2,\ (x,y) \leq (x',y') \Leftrightarrow x \leq x' \text{ and } \ y \leq y'.$$

(1) Show that it is an order relation. Is this order total?

(2) Dztermine $\sup(A)$, $\inf(A)$, $\max(A)$ and $\min(A)$ if they exist, where

$$A = \{(1,2),\ (3,1)\}.$$

Exercise 8

Let S be the relation defined on $\mathbb{R}$ by :

$$\forall x,\ y \in \mathbb{R},\ x\ S\ y \ \Leftrightarrow x^3 - y^3 = x - y$$

(1) Show that S is an equivalence relation.

(2) Discuss according to the value of $m \in \mathbb{R}$, the number of elements of $cl(m)$.