Methods of proof

Contents

Introduction

- In mathematics, proving statements is fundamental.
- Different methods are used to establish the truth of mathematical statements.
- Here are some common methods of proofs :

Direct proof

- A direct proof demonstrates the truth of a proposition by a straightforward chain of logical deductions.
- We want to show that the proposition «$P \Longrightarrow Q$ » is true.
- We assume that P is true and we show that then Q is true.
- This is the method you are most familiar with.

Example 1

- Let n be a natural number.
- Show that : $\quad n$ is even $\Longrightarrow n^2$ is even.
- n is even $\Longrightarrow \exists\ k \in \mathbb{N}$ such that $n = 2k$
- We have $n^2 = (2k)^2 = 2\left(2k^2\right) = 2l$ with $l = 2k^2 \in \mathbb{N}$.
- And consequently n^2 is even.

Example 2

- Show that : $x, y \in \left]-1,1\right[\Longrightarrow \dfrac{x+y}{1+xy} \in \left]-1,1\right[$.
- We have

$$\alpha \in \left]-1,1\right[\iff -1 < \alpha < 1 \iff |\alpha| < 1 \iff \alpha^2 < 1 \iff \alpha^2 - 1 < 0$$

- Let $x, y \in \left]-1,1\right[$.

$$\left(\frac{x+y}{1+xy}\right)^2 - 1 = \frac{(x+y)^2 - (1+xy)^2}{(1+xy)^2} = \frac{x^2+y^2-1-x^2y^2}{(1+xy)^2}$$
$$= \frac{x^2 - 1 + y^2\left(1-x^2\right)}{(1+xy)^2} = \frac{\left(x^2-1\right)\left(1-y^2\right)}{(1+xy)^2} < 0$$

- because $x, y \in \left]-1,1\right[$, and $x^2 < 1,\ y^2 < 1$
- then $\left(\dfrac{x+y}{1+xy}\right)^2 - 1 < 0 \Longrightarrow \left(\dfrac{x+y}{1+xy}\right)^2 < 1 \Longrightarrow \left|\dfrac{x+y}{1+xy}\right| < 1$
- Finally : $\left(\dfrac{x+y}{1+xy}\right)^2 - 1 < 0 \Longrightarrow \dfrac{x+y}{1+xy} \in \left]-1,1\right[$.

Disjunction of cases

- If we want to check a proposition $P(x)$ for all x in a set E,
- we show the proposition for the x in a part A of E,
- then for the x not belonging to A.
- This is the method of disjunction of cases or case by case.

Example 1

- Let $n \in \mathbb{N}$. Show that : $n(n+1)(n+2)$ is even.

<u>*First case* : n is even</u> $\exists\, k \in \mathbb{N}$ such that $n = 2k$
$n(n+1)(n+2) = 2k(2k+1)(2k+2) = 2l$ with
$l = k(2k+1)(2k+2) \in \mathbb{N}$

- therefore $n(n+1)(n+2)$ is even.

<u>*Second case* : n is odd</u> $\exists\, k \in \mathbb{N}$ such that $n = 2k+1$

$$\begin{aligned} n(n+1)(n+2) &= (2k+1)(2k+2)(2k+3) \\ &= 2(2k+1)(k+1)(2k+3) \\ &= 2l \text{ with } l = (2k+1)(k+1)(2k+3) \in \mathbb{N}. \end{aligned}$$

- So, $n(n+1)(n+2)$ is even.
- Conclusion : $\forall n \in \mathbb{N}$, $n(n+1)(n+2)$ is even.

Example 2

- Show that :

$$\forall x \in \mathbb{R}, |x-2| \leq x^2 - 3x + 3$$

- We have $|x-2| = \begin{cases} x-2 \text{ if } x \geq 2 \\ 2-x \text{ if } x < 2 \end{cases}$
-
$$\text{If } x \geq 2, |x-2| = x-2 \leq x^2 - 3x + 3$$
-
$$\text{then } x^2 - 4x + 5 \geq 0 \text{ it's true because } \triangle = -4 < 0$$
-
$$\text{If } x < 2, |x-2| = 2-x \leq x^2 - 3x + 3$$
-
$$\text{then } x^2 - 2x + 1 = (x-1)^2 \geq 0 \text{ it is true.}$$
- Conclusion :

$$\forall x \in \mathbb{R}, |x-2| \leq x^2 - 3x + 3$$

Contraposition

- A proof by contrapositive proves an implication $P \implies Q$ by proving the equivalent contrapositive statement $\overline{Q} \implies \overline{P}$.
- Proof by contraposition is based on the following equivalence

$$(P \implies Q) \iff (\overline{Q} \implies \overline{P})$$

- So, if we want to show the proposition $P \implies Q$
- We actually show that if $\overline{Q}$ is true then $\overline{P}$ is true.

Example 1

- Let $n \in \mathbb{N}$.
- Show that : n^2 is even $\Longrightarrow n$ is even.
- n is odd $\Longrightarrow \exists\ k \in \mathbb{N}$ shuch that $n = 2k + 1$;
- We have

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2\left(2k^2 + 2k\right) + 1 = 2l + 1$$

- with $l = 2k^2 + 2k \in \mathbb{N}$.
- And consequently n^2 is odd .
- Conclusion : n^2 is even $\Longrightarrow n$ is even.

Example 2

- Let $x, y \in \mathbb{R}$.
- Show that :

$$x \neq y \text{ and } xy \neq 1 \Longrightarrow \frac{x}{x^2+x+1} \neq \frac{y}{y^2+y+1}$$

Example 2

- $\dfrac{x}{x^2+x+1} = \dfrac{y}{y^2+y+1} \Longrightarrow x\left(y^2+y+1\right) = y\left(x^2+x+1\right)$

Contradiction

- A proof by contradiction assumes that the statement to be proven is false and then shows that this assumption leads to a contradiction.
- Let R be a proposition. We know that $R \vee \overline{R}$ is true.
- To show that R is true, we assume that R is false, that is to say $\overline{R}$ is true and we show that we obtain a contradiction.
- If R is an implication, $R \approxeq P \Longrightarrow Q$
- We have $\overline{P \Longrightarrow Q} \Longleftrightarrow P \wedge \overline{Q}$
- Proof by contradiction to show that $P \Longrightarrow Q$ is based on the following principle :
- we assume both that P is true and that Q is false and we look for a contradiction.
- So if P is true then Q must be true and therefore $P \Longrightarrow Q$ is true.

Example 1

- Let show by contradiction that $\sqrt{2}$ is irrational $\left(\text{not rational}\right)$:
- We assume that $\sqrt{2} \in \mathbb{Q}$.
- $\sqrt{2} \in \mathbb{Q} \Longrightarrow \sqrt{2} = \frac{a}{b}$ with a and b are natural numbers that are prime to each other. $\left(\text{the fraction} \frac{a}{b} \text{is irreducible}\right)$.

$\Longrightarrow$

Example 2

- Let $a, b > 0$.
- Show that :

$$\frac{a}{1+b} = \frac{b}{1+a} \implies a = b$$

Example 2

- Assume that $\dfrac{a}{1+b} = \dfrac{b}{1+a}$ with $a \neq b$
- $\dfrac{a}{1+b} = \dfrac{b}{1+a} \Longrightarrow a\,(1+b) = b\,(1+a)$

  $\Longrightarrow$