1- Logic and proofs

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- Propositional logic
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1-1 Proposition (statement)

- Mathematical logic allows the study of mathematics as a language.
- Mathematical logic is essential for the statement of a proposition and the study its truth value. So, this is the basis of all mathematical reasonning.
- Logic and proofs form the foundation of mathematics.
- In this course, we will explore the basic concepts of logic, the structure of mathematical proofs, and various proof techniques.

1-1 Propositional logic

- Propositional logic deals with propositions and their logical relationships.
- **Definition**
- A proposition (statement) is a mathematically precise statement that is either true or false, but not both.
- We often note a proposition by letters P, Q, R, ...
- If a proposition P is true, it is assigned the value 1 or T (true), and if it is false, it is assigned the value 0 or F (false).
- $P : \begin{cases} \text{true} & \longrightarrow 1 \text{ or } T \\ \text{false} & \longrightarrow 0 \text{ or } F \end{cases}$
- Truth table

P
1
0

or

P
T
F

1-1 Propositional logic

Principle of non-contradiction
A proposition can not be true and false at the same time.
Principle of excluded third party
A proposition is either true or false but not a third possibility
Examples

- 374 is divisible by 11 (Proposition-true) : $374 = 34.11$
- The natural number 4 is less than $(<)$ the real number π (Proposition-false) : $\pi \simeq 3,14$.
- $1 + \sqrt{2}$ is not a proposition, because this expression doest not have a true value.
- $x + 1 > 5$ is not a proposition. The true value of this statement relies on what the variable x is assigned.
- It gets a logical statement (proposition) if we choose a value for x.(It is called a propositional function or predicate).

Definition

Wen a proposition depnds on a variable or sereval variables, it is called a propositional function or predicate.

Examples

- $P(x) : e^x \geq 1.$

The predicate $P(x)$ is true if $x \geq 0$ and it is false if $x < 0$.

- $Q(x, y)$: For all real number x, there exists a real number y such that $y > x$.

This predicate is true, because for any real number x, we can choose $y = x + 1$. So that $y = x + 1 > x$.

- $R(x, y)$: There exists a real number x such that, for all real number y, we have $y > x$.

This predicate is not true, because is not possible to find a real number x such that all other real numbers y
are strictly greater than x. There is no smallest real number, because real numbers extend toward negative

- We are particulary interested in combining propositions by operators or connectors (connectives).

Definition

A coumpound proposition is a statement obtained by combining propositions with logical connevtives (operators).

1-2-1 Negation

The negation of a proposition P is denoted by **not(P) or** $\urcorner P$ or $\overline{P}$.

- **not(P)** is true if P is false and alse if P is true.
- Truth table

P	$\overline{P}$
1	0
0	1

or

P	$\overline{P}$
T	F
F	T

Remark

- **not(not(P)) is** P $\overline{(\overline{P})}$ is P. That is the negation of the negation of the proposition P is P.

Examples

- $P : |x| < 1$, its negation is $\overline{P} : |x| \geq 1$.
- Q : 4 is **even**. $\overline{Q}$: 4 is not even. that is to say : 4 is **odd**.
- R : All students are in the lecture hall.
  $\overline{R}$: Not all students are in the lecture hall.
  That is to say $\overline{R}$: There is a student that is not in the lecture hall.
- S : 3 divides 15 **and** divides 81.

$\overline{S}$: 3 does not divide 15 **or** does not divide 81.

- T : If a natural number n is a multiple of 4 then it is even.

$\overline{T}$: A natural number n is a multiple of 4 and it is odd.

1-2-2 Equivalence $\iff$

- $P \iff Q$ is the proposition " P is equivalent to Q ", or " P if and only if Q ".

$P \iff Q$ is true when P and Q are both true or both false.

- Truth table

P	Q	$P \iff Q$
1	1	1
1	0	0
0	1	0
0	0	1

or

P	Q	$P \iff Q$
T	T	T
T	F	F
F	T	F
F	F	T

- Two propositions are equivalent if they have identical truth tables.

Examples

- For a, b two real numbers, $(a.b = 0) \iff (a = 0 \text{ or } b = 0)$.
- For a natural integer n, $(n \text{ is even}) \iff (n^2 \text{ is even})$.
- For real numbers a, b and c with $c \neq 0$,
  $\left(\text{The equation } ax^2 + bx + c = 0 \text{ admits real solutions}\right) \iff (\text{its discriminant } \triangle = b^2 - 4ac \geq 0)$.

1-2-3 Conjunction $\wedge$ " and "

- $P \wedge Q$ is the proposition " P **and** Q ".
  This time for $P \wedge Q$ to be true, we need both P and Q to be true (false otherwise).
  The conjunction of two propositions is false if at least one of these propositions is false or both are false.
- Truth table

P	Q	$P \wedge Q$
1	1	1
1	0	0
0	1	0
0	0	0

or

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Remark

- $P \wedge \overline{P}$ is false. (Principle of non-contradiction)

P	$\overline{P}$	$P \wedge \overline{P}$
1	0	0
0	1	0

Examples

- P : 3 is prime, Q : 3 divides 12

$P \wedge Q$: (3 is prime) **and** (3 divides 12) . This proposition is true.

- P : n is an even natural number, Q : n is an odd natural number.

$P \wedge Q$: n is an even **and** odd natural number. This proposition is false.

- $\overline{P}$: n is an odd natural number. That is Q.

$P \wedge Q \Longleftrightarrow P \wedge \overline{P}$, it is false.

- $x > -1$ **and** $x < 1$ means $|x| < 1$.
- $P \wedge Q : x \leq 3$ **and** $x \geq 1$
- If $x = 2$ then $P \wedge Q$ is true.
- If $x = 5$ then $P \wedge Q$ is false.

1-2-4 Disjunction $\vee$ " or "

- $P \vee Q$ is the proposition " P **or** Q ".

$P \vee Q$ is false when both P and Q are false and is true otherwise.

- The disjunction of two propositions is true if at least one of these propositions is true.

- Truth table

P	Q	$P \vee Q$
1	1	1
1	0	1
0	1	1
0	0	0

or

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Remark

- $P \vee \overline{P}$ is true. (Principle of excluded third party)

Examples

- P : 2 is not prime, Q : 2 divides 5.

$P \vee Q$: (2 is not prime) **or** (2 divides 5) . This proposition is false.

- P : n is an even natural number, Q : n is an odd natural number.

$P \vee Q$: n is an even **or** odd natural number. This proposition is true.

- $\overline{P}$: n is an odd natural number. That is Q.

$P \vee Q \Longleftrightarrow P \vee \overline{P}$, it is true.

- $x < -1$ **or** $x > 1$ means $|x| > 1$.
- $P \vee Q : x \leq 2$ **or** $x \geq 5$
- If $x = 1$ then $P \vee Q$ is true.
- If $x = 3$ then $P \vee Q$ is false.

Remark Exclusive or " XOR " $\oplus$

- In everyday language, there is another " **or** " (**exclusive**) .

$P \oplus Q$ is the proposition " P **or** Q ".

- The statement $P \oplus Q$ is true if and only if exactly one of the statements is true.

$P \oplus Q$ is true if **only one** of these propositions is true and false if both are false or true simultaneously.

- Truth table

P	Q	$P \oplus Q$
1	1	0
1	0	1
0	1	1
0	0	0

or

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

Example

The student chooses math **or** computer science and not both.

De Morgan's laws : Negation of $\wedge$ and $\vee$

$\overline{P \wedge Q} \Longleftrightarrow \overline{P} \vee \overline{Q}$ and $\overline{P \vee Q} \Longleftrightarrow \overline{P} \wedge \overline{Q}$

Proof by truth table

P	Q	$\overline{P}$	$\overline{Q}$	$P \wedge Q$	$\overline{P \wedge Q}$	$\overline{P} \vee \overline{Q}$	$P \vee Q$	$\overline{P \vee Q}$	$\overline{P} \wedge \overline{Q}$
1	1	0	0	1	**0**	**0**	1	**0**	**0**
1	0	0	1	0	**1**	**1**	1	**0**	**0**
0	1	1	0	0	**1**	**1**	1	**0**	**0**
0	0	1	1	0	**1**	**1**	0	**1**	**1**

Definition Tautology-Antilogy(contradiction)

- A proposition that is always true is called a tautology.
- A proposition that is always false is called an antilogy or a contradiction.

Examples

- $P \vee \overline{P}$ is a tautology.
- $P \wedge \overline{P}$ is an antilogy or a contradiction.

1-2-5 Implication $\Longrightarrow$ " If...then... "

- It is an essential connective (operator) in mathematics, because it is thanks to it mathematics advances. It ollows us to state news truths.
- $P \Longrightarrow Q$ is the proposition " P **implies** Q " or " **If** P **then** Q ", which is false when P is true and Q is false and true otherwise.
- The mathematical definition of an implication is :

$$[P \Longrightarrow Q] \Longleftrightarrow [\overline{P} \vee Q]$$

- **Truth table**

P	Q	$\overline{P}$	$P \Longrightarrow Q$	$\overline{P} \vee Q$
1	1	0	**1**	**1**
1	0	0	**0**	**0**
0	1	1	**1**	**1**
0	0	1	**1**	**1**

- $P \Longrightarrow Q$ and $\overline{P} \vee Q$ have identical truth tables.
- P is a suffisant condition for Q.
- Q is a necessary condition for P.

Examples

- The implication $(1 = 2 \Longrightarrow 3 = 4)$ is true.
- (Because if we assume that $1 = 2$, then by adding 2 to both sides of this equality we obtain $3 = 4$)
- The implication $[(1 = 2) \text{ and } (4 = 3)] \Longrightarrow [1 + 4 = 2 + 3]$ is true.
- (Because if $a = b$ and $c = d$ then $a + c = b + d$)
- $0 \leq x \leq 100 \Longrightarrow \sqrt{x} \leq 10$. This implication is true (take the square root).
- $\sin x = 0 \Longrightarrow x = 0$ is false (look for $x = 2\pi$ for example);

Remark

$[P \Longleftrightarrow Q] \Longleftrightarrow [P \Longrightarrow Q] \wedge [Q \Longrightarrow P]$

P	Q	$P \Longrightarrow Q$	$Q \Longrightarrow P$	$[P \Longrightarrow Q] \wedge [Q \Longrightarrow P]$	$P \Longleftrightarrow Q$
1	1	1	1	**1**	**1**
1	0	0	1	**0**	**0**
0	1	1	0	**0**	**0**
0	0	1	1	**1**	**1**

Negation of an implication

- We know that $(P \Longrightarrow Q) \Longleftrightarrow (\overline{P} \vee Q)$ (definition of $\Longrightarrow$)

So, $(\overline{P \Longrightarrow Q}) \Longleftrightarrow \left(\overline{\overline{P} \vee Q}\right) \Longleftrightarrow \left(\overline{(\overline{P})} \wedge \overline{Q}\right)$ (De Morgan's laws).

- Hence $\overline{(P \Longrightarrow Q)} \Longleftrightarrow (P \wedge \overline{Q})$

Examples

- Let a and b be two real numbers.
- $R : [(a = 0)\ \text{ or }\ (b = 0)] \Longrightarrow a.b = 0$ is true

$\overline{R} : [(a = 0)\ \text{ or }\ (b = 0)]$ and $a.b \neq 0$ is false.

- $S : a^2 > 0 \Longrightarrow a > 0$ is false ($a = -2 : a^2 = 4 > 0$).

$\overline{S} : (a^2 > 0)$ and ($a \leq 0$) is true.

- $T : n$ is odd $\Longrightarrow n^2$ is odd. (n is a natural integer) is true.

$\overline{T} : (n$ is odd) and (n^2 is even) is false.

Convese of an implication

- The implication " $Q \Longrightarrow P$ " is called the converse of " $P \Longrightarrow Q$ ".

P	Q	$P \Longrightarrow Q$	$Q \Longrightarrow P$
1	1	1	1
1	0	**0**	**1**
0	1	**1**	**0**
0	0	1	1

- $Q \Longrightarrow P$ is not equivalent to $P \Longrightarrow Q$

Example

- Let x a real number.
- $x > 5 \Longrightarrow x > 1$ is true, but
- $x > 1 \Longrightarrow x > 5$ is false (for example $x = 2$)
- $x^2 > 4 \Longrightarrow x > 2$ is false, because for $x = -3$ we have $(-3)^2 = 9 > 4$ and $x = -3 < 2$

But, $x > 2 \Longrightarrow x^2 > 4$ is true.

Contrapositive of an implication

- Let P and Q be two propositions.
- " $\overline{Q} \Longrightarrow \overline{P}$ " is called the contrapositive of " $P \Longrightarrow Q$ ".
-

P	Q	$\overline{P}$	$\overline{Q}$	$P \Longrightarrow Q$	$\overline{Q} \Longrightarrow \overline{P}$
1	1	0	0	**1**	**1**
1	0	0	1	**0**	**0**
0	1	1	0	**1**	**1**
0	0	1	1	**1**	**1**

- $P \Longrightarrow Q$ is equivalent to its contrapositive.

$$(\overline{Q} \Longrightarrow \overline{P}) \Longleftrightarrow (P \Longrightarrow Q)$$

Examples

- Let a, b be two real numbers and n is a natural integer.
- $[(a = 0 \textbf{ or } b = 0) \Longrightarrow (a.b = 0)] \Longleftrightarrow$
  $[(a.b \neq 0) \Longrightarrow (a \neq 0 \textbf{ and } b \neq 0)]$
- $[(n \neq 2 \textbf{ and } n \text{ is prime}) \Longrightarrow n \text{ is odd}] \Longleftrightarrow$
  $[n \text{ is even} \Longrightarrow (n = 2 \textbf{ or } n \text{ is not prime})]$

1-2 Logical connectives

To remember

- $\overline{P \wedge Q} \iff \overline{P} \vee \overline{Q}$ (De Morgan's laws).
- $\overline{P \vee Q} \iff \overline{P} \wedge \overline{Q}$ (De Morgan's laws).
- $(P \implies Q) \iff (\overline{P} \vee Q)$ (definition of $\implies$)
- $\overline{(P \implies Q)} \iff (P \wedge \overline{Q})$ (negation of $\implies$)
- $(\overline{Q} \implies \overline{P}) \iff (P \implies Q)$ (contrapositive of $\implies$)
- $(Q \implies P)$ **is not equivalent to** $(P \implies Q)$ (conserve of $\implies$)
- $[P \iff Q] \iff [P \implies Q] \wedge [Q \implies P]$

1-2 Logical connectives

Properties of logical connectives

- $\overline{(\overline{P})} \iff P, \quad P \wedge Q \iff Q \wedge P, \quad P \vee Q \iff Q \vee P$
- $(P \iff Q) \iff (Q \iff P)\,, \; P \wedge P \iff P, \quad P \vee P \iff P$
- $(P \wedge Q) \wedge R \iff P \wedge (Q \wedge R)$
- $(P \vee Q) \vee R \iff P \vee (Q \vee R)$
- $P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R)$
- $P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R)$
- $\overline{P \wedge Q} \iff \overline{P} \vee \overline{Q}$
- $\overline{P \vee Q} \iff \overline{P} \wedge \overline{Q}$
- $(P \implies Q) \iff (\overline{P} \vee Q)$
- $\overline{(P \implies Q)} \iff P \wedge \overline{Q}$
- $(\overline{Q} \implies \overline{P}) \iff (P \implies Q)$
- $[P \iff Q] \iff [P \implies Q] \wedge [Q \implies P]$
- $[(P \implies Q) \wedge (Q \implies R)] \implies (P \implies R)$
- $P \wedge \overline{P}$ is false, $P \wedge F$ is false where F is false.
- $P \vee \overline{P}$ is true, $P \vee T$ is true where T is true.

1-3 Mathematical quantifiers

- In mathematics, we often use expressions of the form :

" **for all** ", " **for any** "," **there exists at least** ", " **there exists a unique** ".

- These expressions are called " **quantifiers**".
- The word **quantifier** comes from the word **quantity.**

Definition

There are two types of **quantifiers :**

Universal quantifier $\forall$

- $\forall x \longrightarrow$ for all x
- $\forall x,\ P(x)$: means that the predicate $P(x)$ is true **for all** possible values of x.

Existential quantifier $\exists$

- $\exists x \longrightarrow$ there exists x (there exists at least x) or there is x.
- $\exists x,\ P(x)$: means **there exists** x where $P(x)$ is true.
- Sometimes, we will use also $\exists!\ x,\ P(x)$
- it means there exists a unique x where $P(x)$ is true

Negation of quantifiers

- Consider the universal statement $\forall x,\ P(x)$
- This asserts that $P(x)$ is true for all values of x.
- Hence, if it is false, then this means that there exists at least x such that $P(x)$ is false.
- Similary, the existential statement $\exists x,\ P(x)$, asserts that there exists at least x where $P(x)$ is true.
- Hence, if it is false, this means that for all values of x, $P(x)$ is false. That is $\overline{P(x)}$ is true.
- Therefore, we have the following :

$$\overline{\forall x, P(x)} \iff \exists x, \overline{P(x)}$$
$$\overline{\exists x, P(x)} \iff \forall x, \overline{P(x)}$$

Examples

- $\forall x \in \mathbb{R},\ x^2 \geq 0$ true
- For all real number x, his square is greater than or equal to zero.
- Negation of this proposition is
- $\exists x \in \mathbb{R},\ x^2 < 0$ false
- There exists a real number x, whose his square is less than to zero.
- $\exists!\ n \in \mathbb{N}$ such that $n < 1$ true $(n = 0)$
- There exists a unique natural number n, which is less than one.

Remark 1

- Some statements involve several quantifiers.
- The statement : $\forall x \in \mathbb{R} \in \exists y \in \mathbb{R},\ y > x.\ \ (\textit{true})$

means that for all real number x, there exists at least one real number y, which is greater than x.

- This statement is true (For $x \in \mathbb{R}$ we can choose $y = x + 1 > x$) .
- The order of the quantifiers is very important.
- The statement : $\exists y \in \mathbb{R}, \forall x \in \mathbb{R},\ y > x.\ \ (\textit{false})$.
- There does not exist a real number which is greater than all other real numbers.

Remark 2

- $\forall x, \exists y,\ P(x,y) \nLeftrightarrow \exists y, \forall x,\ P(x,y)$

  is not equivalent to

- $\exists x, \forall\ y,\ P(x,y) \nLeftrightarrow \forall y, \exists x,\ P(x,y)$

  is not equivalent to

- $\forall x, \forall y,\ P(x,y) \Longleftrightarrow \forall y, \forall x,\ P(x,y)$
- $\exists x,\ \exists y,\ P(x,y) \Longleftrightarrow \exists y,\ \exists x,\ P(x,y)$
- $\overline{\forall x, \exists y, P(x,y)} \Longleftrightarrow \exists x, \forall y,\ \overline{P(x,y)}$
- $\overline{\exists x, \forall y, P(x,y)} \Longleftrightarrow \forall x, \exists y,\ \overline{P(x,y)}$

Negation of there exists a unique

- Negation of $\exists!\ x \in E$

$$\left[\exists!\ x \in E, P(x)\right] \Longleftrightarrow$$
$$\underbrace{\left[\exists x \in E, P(x)\right]}_{Existence} \textbf{ and } \underbrace{\left[\forall x, x' \in E, \left(P(x)\ \ and\ \ P(x') \Longrightarrow x = x'\right)\right]}_{uniqueness}$$

- Then

$$\overline{\left[\exists!\ x \in E, P(x)\right]} \Longleftrightarrow$$
$$\overline{\left[\exists x \in E, P(x)\right]} \textbf{or} \overline{\left[\forall x, x' \in E, \left(P(x)\ ,\ P(x') \Longrightarrow x = x'\right)\right]}$$
$$\overline{\left[\exists!\ x \in E, P(x)\right]} \Longleftrightarrow$$
$$\left[\forall x \in E, \overline{P(x)}\right] \textbf{ or } \left[\exists x, x' \in E, \left(P(x)\ ,\ \ P(x')\ \ and\ x \neq x'\right)\right]$$

Example

- $\exists!\ x \in \mathbb{R}, \ln x = 1$ is **true** $(\ln e = 1$ and $x = e$ is unique$)$
- $\overline{[\exists!\ x \in \mathbb{R}, \ln x = 1]} \Longleftrightarrow$

$[\forall x \in \mathbb{R}, \ln x \neq 1]$ **or** $[\exists x, x' \in \mathbb{R}, (\ln x = 1 = \ln x' = 1$ **and** $x \neq x')]$

FALSE **FALSE**

- That is $\overline{[\exists!\ x \in \mathbb{R}, \ln x = 1]}$ is **FALSE**