Tutorial Sheet number 02

Exercise 1. Write the following matrices in tabular form indicating their sizes:

a) $A_1 = (2i - j)_{2,3}$.

b) $A_2 = (\delta_{i,j})_{\substack{1\leq i\leq 4\\ 1\leq j\leq 3}}$, where $\delta_{i;j} = \begin{cases} 1 & \text{if } i \leq j, \\ 0 & \text{else.} \end{cases}$

c) **Optional.** $A_3 = ((-1)^{i+j})_{5,2}$.

d) **Optional.** $A_4 = \left(j \sin\left((-1)^i \frac{\pi}{i}\right)\right)_{3,4}$.

Exercise 2. Given $A = \begin{pmatrix} a & b \\ 2 & a^2 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & b/3 \\ b & -a \end{pmatrix}$ where $a, b \in \mathbb{R}$. Find the values of a and b if $A - 3B = \begin{pmatrix} 2 & 0 \\ 1 & -2 \end{pmatrix}$.

Exercise 3. Given the following matrices, find the indicated products, if possible.

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \end{pmatrix},\ B = \begin{pmatrix} 2 & 1 \\ -2 & 0 \end{pmatrix},\ C \begin{pmatrix} -1 & 0 & 2 \\ 0 & -2 & 3 \\ 2 & 1 & -3 \end{pmatrix},\ \mathbf{u} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix},\ \mathbf{v} = \begin{pmatrix} 2 \\ -3 \end{pmatrix},\ \mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

(a) $\mathbf{u}^T\,\mathbf{w}$

(b) $A\,B$

(c) $B\,A$

(d) $A^T\,B$

(e) $A^T\,\mathbf{u}$

(f) **Optional.** $A\,C$

(g) **Optional.** $C\,A$

(h) **Optional.** $C\,\mathbf{u}$

(i) **Optional.** $C\,\mathbf{w}$

(j) **Optional.** $\mathbf{u}^T\,\mathbf{v}$

(k) **Optional.** $\mathbf{u}\,\mathbf{w}^T$

(l) **Optional.** $\mathbf{v}^T\,A$

Exercise 4. Let A and B be two $n \times n$ invertible matrices ($n \in \mathbb{N}$). Simplify the following expressions

1) $A\,(-B + A) + (2B - A + BA^{-1})\,A - B\,(A + I_n + (A - I_n)B^{-1})$.
2) **Optional.** $(2I_n + B^{-1})\,B + A\,(B - A^{-1} + 2A^{-1}B) - AB$.
3) **Optional.** $A\,(7I_n + A^{-1}) - B\,(5B^{-1}A + B^{-1} + A) + A\,(B - 2I_n)$.

Exercise 5. Let $A = \begin{pmatrix} -1 & -2 \\ 3 & 4 \end{pmatrix}$.

1. Compute A^2 and $A^2 - 3A + 2I_2$.
2. Deduce that the matrix A is invertible and determine its inverse.

Exercise 6. Optional. Let $A = \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}$.

1. Perfom A^2 and find the real number α, s.t., $A^2 + \alpha I_2 = 0$.
2. Deduce that the matrix A is invertible and determine its inverse.

Exercise 7. Optional. Let $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix}$.

1. Calculate $A^3 - A$.
2. Deduce that the matrix A is invertible and estimate its inverse.

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Exercise 8. Given the matrix

$$A = \begin{pmatrix} 2 & 3 & 0 \\ -1 & 2 & 4 \\ 0 & 5 & 1 \end{pmatrix}.$$

1. Find the submatrices $A_{1,2}$ and $A_{2,2}$.
2. Calculate the minors $|A_{1,2}|$ and $|A_{2,2}|$.
3. Calculate the cofactors of the elements $a_{1,1}, a_{2,1}, a_{3,1}$ of matrix A.
4. Evaluate the determinant of matrix A.
5. Is A invertible, if yes, estimate its inverse.

Exercise 9. Using elementary row operations, find inverses for the following matrices, if possible.

a. $\begin{pmatrix} 2 & 1 & 2 \\ 0 & 3 & -1 \\ 4 & 1 & 1 \end{pmatrix}$.

b. **Optional.** $\begin{pmatrix} 3 & -1 & 5 \\ -1 & 2 & 1 \\ -2 & 4 & 3 \end{pmatrix}$.

c. **Optional.** $\begin{pmatrix} 2 & 4 & 3 \\ -1 & 3 & 0 \\ 0 & 2 & 1 \end{pmatrix}$.

d. **Optional.** $\begin{pmatrix} 1 & 1 & 2 \\ 2 & -3 & 1 \\ 4 & -1 & 5 \end{pmatrix}$.

e. **Optional.** $\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 2 & 7 \end{pmatrix}$.

f. **Optional.** $\begin{pmatrix} 2 & 4 & 3 \\ -1 & 3 & 0 \\ 0 & 2 & 1 \end{pmatrix}$.

g. **Optional.** $\begin{pmatrix} -1 & 5 & 3 \\ 4 & 0 & 0 \\ 2 & 7 & 8 \end{pmatrix}$.

h. **Optional.** $\begin{pmatrix} 2 & -3 & 1 \\ 3 & 4 & -2 \\ 5 & 1 & -1 \end{pmatrix}$.

i. **Optional.** $\begin{pmatrix} 3 & 1 & 2 \\ 4 & 5 & 1 \\ -1 & 2 & -1 \end{pmatrix}$.

Optional. Using comatrix (adjoint matrix), find inverses for the previous matrices, if possible.

Exercise 10. Find the solution of the following systems of equations by Cramer's rule and by Gaussian elimination:

$$\begin{cases} 2x_1 + 4x_2 + 3x_3 &= 1 \\ 3x_1 - 6x_2 - 2x_3 &= -2 \\ -5x_1 + 8x_2 + 2x_3 &= 4 \end{cases}$$

Exercise 11. Optional. For all $m \in \mathbb{R}$, we consider the following system

$$\begin{cases} x_1 + 2x_3 &= 4 \\ 2x_1 + mx_2 + 4x_3 &= 8 - m \\ -x_1 - mx_2 + \left(m^2 - 3m - 2\right) x_3 &= 4 \end{cases}$$

Solve the system above according to the value of the parameter m.

Exercise 12. Find the characteristic polynomial, eigenvalues, and an invertible matrix Q such that $Q^{-1}AQ$ is a diagonal matrix if

- $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$
- **Optional.** $A = \begin{pmatrix} 3 & 1 & 1 \\ -4 & -2 & -5 \\ 2 & 2 & 5 \end{pmatrix}$