UNIVERSITY OF TLEMCEN
FACULTY OF SCIENCES
First-year LMD-SM
Mathematics 2

TUTORIAL SHEET NUMBER 01

Exercise 1. Let f be a function defined on $\mathbb{R}$ by $f(x) = \frac{\sin x + \cos x}{1+\cos^2 x}$. Show that, for all $a \in \mathbb{R}$, f' has at least a zero on each interval $(a; a+2\pi)$.

Exercise 2. Optional. Let p and q be two real numbers and n be a natural number greater than or equal to 2. Show that the polynomial P defined on $\mathbb{R}$ by $P(x) = x^n + px + q$ has at most three real roots if n is odd and, at most two real roots if n is even.

Exercise 3.

1. Using the Mean Value Theorem, show that:
$$\forall x \in \mathbb{R}, \qquad \frac{1}{1+x} < \ln(x+1) - \ln x < \frac{1}{x}.$$
2. Deduce that the functions f and g defined on $\mathbb{R}_+^*$ by $f(x) = \left(1+\frac{1}{x}\right)^x$ and $g(x) = \left(1+\frac{1}{x}\right)^{x+1}$ are monotonic.
3. Determine the limit at infinity of $\ln f$ and $\ln g$, then f and g.

Exercise 4. Let $n \in \mathbb{N}^*$. Establish the n^{th} derivative of the following functions

(i) $f_1 : x \mapsto f_1(x) = \exp(ax), \quad a \in \mathbb{R}^*$.

(ii) $f_2 : x \mapsto f_2(x) = \sin(x)$.

(iii) $f_3 : x \mapsto f_3(x) = \frac{1}{1+x}$.

(iv) **Optional.** $f_4 : x \mapsto f_4(x) = \frac{1}{1-x}$.

Exercise 5.

1. Compute the asymptotic (power) expansion of order three of the following functions in a neighborhood of 0:
   (a) $x \mapsto \tan x$.
   (b) $x \mapsto \ln^2(1+x)$.
   (c) $x \mapsto \frac{1}{\cos x}$.
   (d) **Optional.** $x \mapsto \exp(\cos x)$.
   (e) **Optional.** $x \mapsto (1+x)^{1/(1+x)}$.
   (f) **Optional.** $x \mapsto \frac{\sqrt{1+x^2}}{1+x+\sqrt{1+x^2}}$.
2. **Optional.** Deduce simple asymptotic equivalences of these functions in a neighborhood of 0.

Exercise 6. Considering the functions f_i, $(i = \overline{1,5})$ given for all $x > 0$ by

- $f_1(x) = x \ln x$,
- $f_2(x) = \ln(1+x)$,
- $f_3(x) = x^2 \exp\left(\frac{1}{x}\right)$,
- $f_4(x) = \frac{x^3}{\ln x}$,
- $f_5(x) = \exp\left(\frac{1}{\ln x}\right)$.

1. Verify that in a neighborhood of 0, we have: $f_4 = o(f_2), \quad f_2 = o(f_1), \quad f_1 = o(f_5), \quad f_5 = o(f_3)$.
2. **Optional.** Verify that in a neighborhood of ∞, we have: $f_5 = o(f_2)$, $f_2 = o(f_1)$, $f_1 = o(f_3)$, $f_3 = o(f_4)$

Exercise 7. Estimate the following limits

1. $\lim\limits_{x\to 0} \frac{\exp(x^2) - \cos x}{x^2}$.
2. $\lim\limits_{x\to 0} \frac{\ln(1+x) - \sin x}{x}$.
3. $\lim\limits_{x\to 0} \frac{\cos x - \sqrt{1-x^2}}{x^4}$.
4. **Optional.** $\lim\limits_{x\to -\infty} \sqrt{x^2+3x+2} + x$.
5. **Optional.** $\lim\limits_{x\to +\infty} \left(\cos\frac{1}{x}\right)^{x \ln x}$.
6. **Optional.** $\lim\limits_{x\to 1} \frac{(2x-x^3)^{1/3} - \sqrt{x}}{1-x^{3/4}}$.