Tutorial Sheet number 03

Exercise 1. Establish the following limits when they exist

(i) $\lim_{x\to+\infty} \frac{x^2+2x-1}{x^2+x-3}$.

(ii) $\lim_{x\to 0} \frac{x^2+2|x|}{x}$.

(iii) $\lim_{x\to 1} \frac{1}{1-x} - \frac{2}{1-x^2}$.

(iv) $\lim_{x\to+\infty} \sqrt{x-1} - \sqrt{x-4}$.

(v) $\lim_{x\to+\infty} \left(\frac{1}{x}\right)^{x^2}$.

(vi) $\lim_{x\to+\infty} \frac{\cos x}{x^2}$.

(vii) $\lim_{x\to 0} \frac{x}{a}\left\lfloor \frac{b}{x} \right\rfloor$, $\quad a,\ b \in \mathbb{R}_+^*$.

(viii) **Optional.** $\lim_{x\to 0} \frac{b}{x}\left\lfloor \frac{x}{a} \right\rfloor$, $\quad a,\ b \in \mathbb{R}_+^*$.

Exercise 2. Optional. Let f be the function defined on $\mathscr{D} \subseteq \mathbb{R}$ by $f(x) = \left\lfloor \frac{1}{x} \right\rfloor$.

1. Determine the domain $\mathscr{D}$ of f.
2. Calculate $\lim_{x\to 0^+} f(x)$.
3. Deduce $\lim_{x\to 0^+} x\left\lfloor \frac{1}{x} \right\rfloor$ and $\lim_{x\to 0^+} x^2\left\lfloor \frac{1}{x} \right\rfloor$.

Exercise 3. Let f be the floor function, i.e., $f : x \mapsto f(x) = \lfloor x \rfloor$.

1. Plot the graph of the function f.
2. At which points f is continuous ?
3. Among its points of discontinuity, at which points is it continuous on the left? continuous on the right?
4. Is the function $g : x \mapsto g(x) = \lfloor x \rfloor + (x - \lfloor x \rfloor)^2$ continuous.

Exercise 4. Optional. Let f be a contiuous function on $\mathbb{R}$. Show that if f is a zero function on $\mathbb{Q}$, then f is a constant zero function.

Exercise 5. Let f and g defined by

$$f: \begin{array}{ccc} \mathbb{R}_+^* & \longrightarrow & \mathbb{R} \\ x & \mapsto & f(x) = \dfrac{\sin x}{\sqrt{x}}. \end{array} \qquad g: \begin{array}{ccc} \mathbb{R}_+^* & \longrightarrow & \mathbb{R} \\ x & \mapsto & g(x) = (x)^{1/x}. \end{array}$$

1. Are f and g continuous.
2. Establish the continuous extension of f on $\mathbb{R}_+$.
3. Has g a continuous extension at 0 ?

Exercise 6. Optional. Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic continuous function. Show that, if f has a limit at infinity, then f is a constant function.

Exercise 7. Optional. Let f and g be two reel functions defined on $\mathscr{D} \subseteq \mathbb{R}$, $x_0 \in \mathscr{D}$; and $\ell_1,\ \ell_2 \in \mathbb{R}$. Show that, if $\lim_{x\to x_0} f(x) = \ell_1$ and $\lim_{x\to x_0} g(x) = \ell_2$, then

(i) $\lim_{x\to x_0} (f(x) + g(x)) = \ell_1 + \ell_2$;

(ii) $\lim_{x\to x_0} (f(x) \cdot g(x)) = \ell_1 \cdot \ell_2$.