Matrices and Determinants.

University of Tlemcen
Faculty of Sciences

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1 Matrices (Definition, Operation)

1.1 Definitions

- The matrix A of type (m, n), with m rows and n columns, is a rectangular array of elements from the set $\mathbb{K}$.
- The numbers in the array are called the coefficients of A.
- The coefficient located in the i-th row and j-th column is denoted as a_{ij}.
- Such an array is represented as follows:

$$A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1j} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2j} & ... & a_{2n} \\ ... & ... & ... & ... & ... & ... \\ a_{i1} & a_{i2} & ... & a_{ij} & ... & a_{in} \\ ... & ... & ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mj} & ... & a_{mn} \end{pmatrix} \text{ where } A = (a_{ij}) \quad \text{avec} \quad 1 \leq i \leq m \quad \text{et} \quad 1 \leq j \leq n.$$

It is said that the matrix A is of size $m \times n$ (read as "m by n") (respecting the order of reading).

- We denote by $M_{mn}(\mathbb{K})$ the set of matrices of size $m \times n$ with coefficients in $\mathbb{K}$.
  $M_n(\mathbb{K})$ is the set of square matrices of order n with coefficients in $\mathbb{K}$.
- The elements of $M_{mn}(\mathbb{R})$ are called **real matrices**.
- **Two matrices are equal** when they have the same size and corresponding coefficients are equal.

1.2 Special Matrices

1 **Column matrices**" are matrices with one column $\begin{pmatrix} a_{11} \\ a_{22} \\ \vdots \\ a_{m1} \end{pmatrix}$ So of size $m \times 1$

2 **Row matrices**" are matrices with a single row $\begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \end{pmatrix}$.

3 **The zero matrix** is the matrix in which all elements are zero. It is denoted as 0_{mn}.

4 **Square matrices**" are matrices in which the number of rows is equal to the number of columns. This number is called the order of the matrix.

The coefficients with the same row and column indices are called **diagonal coefficients**. Exemple : $a_{11}, a_{22}, a_{33}, ...$

5 **Lower triangular matrices** are square matrices where all coefficients strictly above the diagonal (i.e., indices ij with $j > i$) are zero.

6 **Upper triangular matrices** are square matrices where all coefficients strictly below the diagonal (i.e., indices ij with $j < i$) are zero.

7 **Diagonal matrices** are square matrices that are both upper triangular and lower triangular. The only non-zero elements are those on the diagonal.

8 **The identity matrix** is the diagonal matrix where all diagonal coefficients are 1. It is denoted as I_n, the identity matrix of order n.

9 A square matrix of order n such that $a_{ii} = a \in \mathbb{K},\ 1 \leq i \leq n$, and $a_{ij} = 0$ for all $i \neq j$ is called a **scalar matrix**.

2 Operations on Matrices

Matrix Addition

Sum of Two Matrices: Let $A = (a_{ij})$ with $1 \leq i \leq m$ and $1 \leq j \leq n$ and $B = (b_{ij})$ with $1 \leq i \leq m$ and $1 \leq j \leq n$ be two matrices of the same size $m \times n$. Their sum $C = A + B$ is the matrix of size $m \times n$ defined by:

$$c_{ij} = a_{ij} + b_{ij}$$

Proposition 2.1 *Let A, B, and C be three matrices of $M_{mn}(\mathbb{K})$:*

- *Addition is* ***associative****: $(A + B) + C = A + (B + C)$.*
- *The zero matrix with m rows and n columns is the identity element for addition:*
  $A + 0 = A$.

- *Every matrix A has a **symmetric or 'opposite'** matrix $(-A)$.*
  By defining $A = (a_{ij}) \quad \text{with} \quad 1 \leq i \leq m \quad \text{et} \quad 1 \leq j \leq n$ and $-A = (-a_{ij}) \quad \text{avec} \quad 1 \leq i \leq m \quad \text{et} \quad 1 \leq j \leq n$, we have $A + (-A) = 0$.

- *Addition is **commutative**: $A + B = B + A$.*

- *We denote $A - B$ as the sum $A + (-B)$.*

Product of a matrix by an element of $\mathbb{K}$
Let $A = (a_{ij}) \quad \text{with} \quad 1 \leq i \leq m \quad \text{et} \quad 1 \leq j \leq n$ and $\lambda \in \mathbb{K}$), then $\lambda A = (\lambda a_{ij}) \quad \text{avec} \quad 1 \leq i \leq m \quad \text{et} \quad 1 \leq j \leq n$.

Proposition 2.2 *Let λ and μ be two elements of $\mathbb{K}$, and A and B be two matrices of M_{mn}, then:*

- $\lambda(A + B) = \lambda A + \lambda B$.
- $(\lambda + \mu)A = \lambda A + \mu A$.
- $\lambda(\mu A) = (\lambda\mu)A$.
- $1A = A$.

Product of two matrices:
The product AB of two matrices A and B is defined if and only if the number of columns of A is equal to the number of rows of B.
Let $A = (aij)_{m,n}$ and $B = (bij)_{n,p}$: The product of A and B is a matrix of type (m, p).

$$A \times B = (c_{mp}) \text{ such that } c_{ij} = \sum_{i=1}^{n} a_{ik} b_{kj}$$

Properties

Let $n, m, p, q \in \mathbb{N}^*$

- **Associativité.** Let $A \in M_{mn}$, $B \in M_{np}$ and $C \in M_{pq}$, then

$$(AB)C = A(BC)$$

- **Role of identity matrices** $A \in M_{mn}$:

$$AI_n = A \text{ and } I_m A = A$$

- **Distributivity with respect to addition**
  If A and B are two matrices of $M_{mn}(\mathbb{K})$ and $C \in M_{np}(\mathbb{K})$, then

$$(A + B)C = AC + BC$$

. If $A \in M_{mn}(\mathbb{K})$ and if B and $C \in M_{np}(\mathbb{K})$, then

$$A(B + C) = AB + AC$$

.

- **Compatibility with the outer product.** If $A \in M_{mn}(\mathbb{K})$ and if $B \in M_{np}(\mathbb{K})$, $\lambda \in \mathbb{K}$ then

$$\lambda(AB) = (\lambda A)B = A(\lambda B)$$

- **Matrix power** In the set $M_n(\mathbb{K})$ of square matrices of size $n \times n$ with coefficients in $\mathbb{K}$, matrix multiplication is an internal operation:

$$\text{Si} \quad A, B \in M_n(\mathbb{K}) \quad \text{then,} \quad AB \in M_n(\mathbb{K})$$

. In particular, one can multiply a square matrix by itself, denoted as:

$$A^2 = A \times A, A^3 = A \times A \times A$$

. One can thus define the successive powers of a matrix.

Définition 2.3 *For all $A \in M_n(\mathbb{K})$, successive powers of A are defined by*
$A^0 = I$ *and*
$A^{p+1} = A^p \times A$*, for all* $p \in \mathbb{N}$.
In other words, $A^p = \underbrace{A \times A \times \cdots \times A}_{p \text{ factors}}$

Transpose of a matrix

Let $A = (a_{ij})$ with $1 \leq i \leq m$ and $1 \leq j \leq n$, **the transpose** of A is called, and denoted by ${}^t A$ the matrix with n rows and m columns

$${}^t A = (a_{ji}) \quad \text{with} \quad 1 \leq j \leq n \quad \text{and} \quad 1 \leq i \leq m$$

.

Proposition 2.4 *Let A and B be two matrices in $M_{mn}(\mathbb{K})$ and $\lambda \in \mathbb{K}$. Then,*

- ${}^t({}^t A) = A$
- ${}^t(\lambda A) = \lambda \ {}^t A$
- ${}^t(A + B) = {}^t A + {}^t B$
- ${}^t(AB) = {}^t B \ {}^t A$

Trace of a matrix"

The **trace** of the matrix A is the number obtained by **by adding the diagonal elements of** A. In other words,

$$trA = a_{11} + a_{22} + \cdots + a_{nn}$$

Theorem 2.5 *Let A and B two matrices $n \times n$. Then:*

- $tr(A + B) = trA + trB$

- $tr(\alpha A) = \alpha tr(A)$
- $tr({}^t A) = tr(A)$
- $tr(AB) = tr(BA)$

Définition 2.6 *A matrix A of size $n \times n$,*

- *A is called* ***symmetric****, if it is equal to its transpose. That is, ${}^t A = A$*
- *A is called* ***antisymmetric****, if ${}^t A = -A$*

Inverse of a matrix

Définition 2.7 *Let A a square matrix of size $n \times n$. If there exists a square matrix B of size n such that:*

$$AB = BA = I,$$

it is said that A is ***invertible****.*

- *We call B the inverse of A and denote it A^{-1} .*
  When A is invertible for every $p \in \mathbb{N}$, we denote it as :
  $$A^{-p} = (A^{-1})^p = \underbrace{A^{-1} \times A^{-1} \times \cdots \times A^{-1}}_{p\ factors}$$
- *The set of invertible matrices in $M_n(\mathbb{K})$ is denoted as $GL_n(\mathbb{K})$.*

Proposition 2.8

- *If A is invertible, then its inverse is unique*
- *Let A be an invertible matrix. Then A^{-1} is also invertible and we have:*
  $(A^{-1})^{-1} = A$.
  $(\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}$, $\lambda \in \mathbb{K}$.
  ${}^t(A^{-1}) = ({}^t A)^{-1}$
- *Let A and B be two invertible matrices of the same size. Then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.*
- *Let A, B, and C be three matrices in $M_n(\mathbb{K})$. Then the equality $AC = BC$ implies the equality $A = B$.*

Matrix Inverse: Calculation

We will see a method for efficiently calculating the inverse of any matrix. We begin with a direct formula in the simple case of matrices 2×2.

Matrice 2×2: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Si $ad - bc \neq 0$ alors A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Determinant of a matrix

1) If $A = (a_{11}) = a_{11}$.
The determinant of A is a real or complex number denoted $det(A) = a_{11}$.

2) If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

Expanding along the first column. The determinant of A is a real or complex number denoted

$$\begin{array}{cc} & \begin{array}{cc} + & - \end{array} \\ det(A) \;\; ou & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12} \end{array}$$

.

3) Si $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. The determinant of A is:

$$\begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \end{array} = +a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

.

4) Si $A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{pmatrix} \in M_n(\mathbb{K}),\ n \geq 2$.

$$det(A) = \sum_{i,j=1}^{n} (-1)^{i+j}\ a_{ij}\ det(M_{ij}).$$

Let $M_{ij} \in M_{n-1}(\mathbb{K})$ be the matrix obtained by removing the i-th row and the j-th column from the matrix A.

Properties:

- $det(AB) = det(A)det(B), \quad A,\ B\ \in M_n(\mathbb{K})$.
- $det(A^{-1}) = \frac{1}{det(A)}, \quad det(A) \neq 0$.
- If all elements in a row (column) of a matrix $A \in M_n(\mathbb{K})$ are zero, then, $det(A) = 0$.
- If all elements in a row (column) of the determinant of a matrix $A \in M_n(\mathbb{K})$ are multiplied by a scalar k, then the determinant is multiplied by k.

- If B is obtained from $A \in M_n(\mathbb{K})$ by exchanging two of its rows (columns), then: $det(A) = -det(B)$.
- If two rows (columns) of $A \in M_n(\mathbb{K})$ are identical, then $det(A) = 0$.

Cofactor matrix of a matrix ($n \times n$)

Cofactor matrix of the matrix $A = (a_{ij})_{n,n}$, denoted $com(A)$ and

$$com(A) = ((-1)^{i+j} det(M_{ij})_{n,n}$$

such that $M_{ij} \in M_n(\mathbb{K})$ the matrix obtained by removing the i-th row and the j-th column from matrix A.

Inverse of a matrix ($n \times n$)

The inverse of matrix A, denoted A^{-1} such that $AA^{-1} = A^{-1}A = I$, is:

$$A^{-1} = \frac{1}{det(A)} \; {}^{t}(com(A)).$$

Remarque 2.9 *A is invertible if and only if $det(A) \neq 0$.*

Similar Matrices

Définition 2.10 *Let A and B be two matrices in $M_n(\mathbb{K})$. We say that matrix B is similar to matrix A if there exists an invertible matrix $P \in M_n(\mathbb{K})$ such that*

$$B = P^{-1}AP$$

2.1 Matrix associated with a linear transformation

2.1.1 Linear applications

We say that the application $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is the linear application ($f \in$ Ł$(\mathbb{R}^n, \mathbb{R}^m)$, if $\forall (X, Y) \in (\mathbb{R}^n)^2, \forall \alpha, \beta \in \mathbb{R}$ we have:

$$f(\alpha X + \beta Y) = \alpha f(X) + \beta f(Y).$$

Définition 2.11 *A linear application $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is an expression that transforms a vector $X = (x_1, x_2, \cdots, x_n)$ from $\mathbb{R}^n$ into a vector $Z = (z_1, z_2, \cdots, z_n)$ of $\mathbb{R}^m$ where each component z_i is given by a linear combination of the coordinates x_i.*
That is, there exist constants a_{ij} such that:

$$f : X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \mapsto Z = \begin{pmatrix} Z_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ Z_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ Z_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \\ \vdots \\ Z_m = a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}.$$

2.1.2 Associated matrix

A linear application f from $\mathbb{R}^n$ to $\mathbb{R}^m$ is uniquely determined by a table A with m rows and n columns of coefficients

$$A = \begin{pmatrix} a_{11} & \cdots & \cdots & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & \cdots & \cdots & \cdots & a_{mn} \end{pmatrix}$$

where a_{ij} is the element of A located in the i-th row and j-th column of A.

Définition 2.12 *A is called the matrix of the linear application f, and we write $A = M_f$. (The matrix associated with the linear application f.)*
We remark that

$$f\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & \cdots & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & \cdots & \cdots & \cdots & a_{mn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$= AX, \quad X \in \mathbb{R}^n$$

2.2 Linear application associated with a matrix

Let $A = \begin{pmatrix} a_{11} & \cdots & \cdots & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & \cdots & \cdots & \cdots & a_{mn} \end{pmatrix}$ a given matrix

The linear application associated with the matrix A is defined by

$$f\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & \cdots & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & \cdots & \cdots & \cdots & a_{mn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$