Integrals

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1 Indefinite Integral and its Properties

Let $f(x)$ be a continuous function on a given interval. The indefinite integral of $f(x)$, denoted $\int f(x)\,dx$, is defined as:

$$\int f(x)\,dx = F(x) + C$$

Where $F(x)$ is a function whose derivative is equal to $f(x)$, i.e., $F'(x) = f(x)$, and C is an arbitrary constant.

1.1 Definition of a Primitive

A primitive of a function is a function whose derivative is equal to the given function. Formally, let $f(x)$ be a function defined on an interval I. A function $F(x)$ is a primitive of $f(x)$ on I if its derivative is equal to $f(x)$ for all x in I, i.e., if $F'(x) = f(x)$ for all x in I.

In mathematical notation, this is written as:

$$F'(x) = f(x)$$

A primitive of $f(x)$ is often denoted $\int f(x)\,dx$ and is called the indefinite integral of $f(x)$.

Fundamental Integration Formulas:

Function	Primitive
$\int k\,dx$	$kx + C$ (where k is a constant)
$\int x^n\,dx$	$\frac{1}{n+1}x^{n+1} + C$ (for all $n \neq -1$)
$\int e^x\,dx$	$e^x + C$
$\int \frac{1}{x}\,dx$	$\ln\lvert x\rvert + C$ (for $x \neq 0$)
$\int \sin(x)\,dx$	$-\cos(x) + C$
$\int \cos(x)\,dx$	$\sin(x) + C$
$\int \tan(x)\,dx$	$-\ln\lvert\cos(x)\rvert + C$
$\int \frac{1}{\sqrt{1-x^2}}\,dx$	$\arcsin(x) + C$
$\int \frac{1}{1+x^2}\,dx$	$\arctan(x) + C$
$\int \frac{1}{\cos^2(x)}\,dx$	$\tan(x) + C$
$\int \frac{1}{\sin^2(x)}\,dx$	$-\cot(x) + C$
$\int \frac{1}{\sqrt{a^2-x^2}}\,dx$	$\arcsin\left(\frac{x}{a}\right) + C$ (for $\lvert x\rvert < \lvert a\rvert$)
$\int \frac{1}{a^2+x^2}\,dx$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$
$\int \frac{1}{\sqrt{x^2+a^2}}\,dx$	$\ln\left(x + \sqrt{x^2+a^2}\right) + C$
$\int \frac{1}{x\sqrt{x^2-a^2}}\,dx$	$\frac{1}{a}\ln\left\lvert\frac{x+\sqrt{x^2-a^2}}{a}\right\rvert + C$ (for $x > a$)
$\int \frac{1}{\sqrt{x^2-a^2}}\,dx$	$\cosh^{-1}\left(\frac{x}{a}\right) + C$ (for $x > a$)
$\int \frac{1}{\sqrt{a^2+x^2}}\,dx$	$\cosh^{-1}\left(\frac{x}{a}\right) + C$
$\int \frac{1}{\sqrt{a^2-x^2}}\,dx$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert + C$ (for $\lvert x\rvert < \lvert a\rvert$)

2 Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

where u is a function that you choose to differentiate (and is typically easier to integrate), and dv is a function that you choose to integrate (and is typically harder to integrate).

2.1 Integration by Change of Variables

Here are the steps to perform a change of variables and simplify an integral:

1. Choice of substitution variable: Identify the complex part of the integral and choose an appropriate substitution variable.

2. Calculation of du: Compute du in terms of dx after choosing the substitution variable u.

3. Replace the variable and dx: Replace u in the integral and dx with the expression found for du.

4. Integration with respect to u: Integrate the new integral with respect to u to make it more manageable.

5. Return to the original variable: Re-express the result in terms of the original variable x after integrating with respect to u.
6. Include the integration constant: Don't forget to include the integration constant C at the end of the result.

3 Integration of Rational Functions

Let $f(x)$ be a rational function defined by $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials and $Q(x)$ is not identically zero.

The goal of integrating rational functions is to find a primitive of $f(x)$ using the method of integration by partial fractions. This method involves decomposing $f(x)$ into a sum of simple fractions, and then integrating each term separately.

3.1 Integration by Partial Fractions

To integrate a rational function, we start by decomposing $f(x)$ into partial fractions of the form $\frac{A}{(x-\alpha)}$, where α is a root of $Q(x)$ and A is a constant to be determined.

3.2 Integration of Rational Fractions

After decomposing a rational function into partial fractions, we integrate each term separately to obtain the primitive of $f(x)$.

4 Integration of Exponential and Trigonometric Functions

4.1 Integration of Exponential Functions

The integration of exponential functions is generally straightforward. For example, for e^x, we have:

$$\int e^x \, dx = e^x + C$$

For functions of the form a^x with $a > 0$, we use a change of variable. For example, for a^x:

$$\int a^x \, dx = \frac{1}{\ln(a)} \cdot a^x + C$$

4.2 Integration of Trigonometric Functions

For functions of the form $\int \sin^n(x) \cos^m(x) \, dx$, we use changes of variable with $\sin(x)$ or $\cos(x)$.

For more complex expressions $\int f(\sin(x), \cos(x), \tan(x))\, dx$, we can use $t = \tan\left(\frac{x}{2}\right)$, with the relations:

$$\cos(x) = \frac{1-t^2}{1+t^2}, \quad \sin(x) = \frac{2t}{1+t^2}, \quad \tan(x) = \frac{2t}{1-t^2}, \quad dx = \frac{2}{1+t^2}\, dt.$$

5 Definite Integration

Définition 5.1 *Let $a < b \in \mathbb{R}$ and f be a continuous function on $[a, b]$, with F being a primitive of f. Then:*

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

is called the definite integral from a to b.

The result of a definite integration is a constant number, not a function.