Linear Systems of Equations

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1 Linear Systems of Equations

Definition 1.1 *A system of linear equations with m equations in n unknowns is a system of the form:*

$$(S_1) : \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1, \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n = b_m. \end{cases}$$

1.1 Matrix Representation

The system (S_1) can be written in matrix form $AX = B$, where:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

1.2 Rank of a Linear System

The rank of a system is the rank of matrix A.

2 Study of the Solution Set

2.1 Characteristic Determinant

The characteristic determinant of a system is a determinant formed from the coefficients of the system.

2.2 Study of the Solution Set

1. If $r = m = n$, the system has a unique solution.
2. If $r < m < n$, the system is indeterminate.

3. If $r < m$ and at least one characteristic determinant is nonzero, the system has no solution.
4. If $r < m$ and all characteristic determinants are zero, the system is reducible and can be solved.

2.3 Equivalent Systems

Two systems are equivalent if they have the same solution set.

2.4 Row-Echelon Systems

A system is triangular if it has as many unknowns as equations and if all coefficients below the diagonal are zero. It is reduced row-echelon if it is triangular with nonzero diagonal coefficients.

3 Solution Methods

3.1 Substitution Method

Choose an equation, express one unknown in terms of the others, and substitute this expression into the other equations.

3.2 Cramer's Rule

For a square system with a unique solution, determinants of associated matrices can be used.

3.3 Matrix Inversion Method

If the system matrix is invertible, the solution can be found by multiplying by the inverse of the matrix.

3.4 Gauss Elimination Method

By a series of elementary operations, transform the system into a row-echelon or reduced row-echelon system, then solve. These methods can solve any system of linear equations.