Linear Algebra

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January 26, 2024

1 Vectorial Spaces

Définition 1.1 *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and E be a set equipped with two rules laws, one denoted by $\oplus$ (internal) and the other denoted by $\otimes$ (external), defined by:"*

- $$\oplus : E \times E \longrightarrow E$$ $$(x, y) \longrightarrow x \oplus y,$$
- $$\otimes : E \times \mathbb{K} \longrightarrow E$$ $$(x, \alpha) \longrightarrow \alpha \otimes x.$$

E is called a vector space over $\mathbb{K}$, and it is denoted as $\mathbb{K}.v.s$ if and only if:

1. *$u \oplus v = v \oplus u$ (for all $u, v \in E$).*
2. *$u \oplus (v \oplus w) = (u \oplus v) \oplus w$ (for all $u, v, w \in E$).*
3. *There exists a neutral element $0_E \in E$ such that $u \oplus 0_E = u$ for all $u \in E$.*
4. *Every $u \in E$ has a symmetric element u' such that $u \oplus u' = 0_E$. This element u' is noted $-u$.*
5. *$\forall u \in E : 1 \otimes u = u$.*
6. *$\forall u \in E, \forall a, b \in \mathbb{K} : a \otimes (b \otimes u) = (a.b) \otimes u$ (. denotes the usual multiplication).*
7. *$\forall u \in E, \forall a, b \in \mathbb{K} : (a + b) \otimes u = (a \otimes u) \oplus (b \otimes u)$ (+ denotes the usual addition).*
8. *$\forall u, v \in E, \forall a \in \mathbb{K} : a \otimes (u \oplus v) = (a \otimes u) \oplus (a \otimes v)$.*

The elements of E are called **vectors***, and the elements of $\mathbb{K}$ are called* **scalars**

Exemple 1.2 *We equip $\mathbb{R}^2$ with the following two rules of computation:*

$$\forall (x, y), (x', y') \in \mathbb{R}^2 : (x, y) + (x', y') = (x + x', y + y')$$

and

$$\forall (x, y) \in \mathbb{R}^2, \forall \alpha \in \mathbb{R} : \alpha.(x, y) = (\alpha x, \alpha y),$$

where + and · represent the usual addition and multiplication, respectively. Equipped with these two operations, $(\mathbb{R}^2, +, \cdot)$ forms an $\mathbb{R}$-vector space. More generally, for $\mathbb{R}^n$ with the rules:

$$(x_1, x_2, ..., x_n) + (y_1, y_2, ..., y_n) = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$$

and The operation defined by $\alpha \cdot (x_1, x_2, ..., x_n) = (\alpha x_1, \alpha x_2, ..., \alpha x_n)$ constitutes an $\mathbb{R}$-vector space.

Proposition 1.3 *Let $(E, \oplus, \otimes)$ be a $\mathbb{K}$-vector space.*

1. $\forall u \in E$, $0 \otimes u = 0_E$ *(where 0_E is the neutral element of E).*
2. $\forall \alpha \in \mathbb{K} : \alpha \otimes 0_E = 0_E$
3. $\forall \alpha \in \mathbb{K}, \forall u \in E : \alpha \otimes (-u) = (-\alpha) \otimes u = -\alpha \otimes u.$
4. $\forall \alpha \in \mathbb{K}, \forall u \in E : \alpha \otimes u = 0 \Leftrightarrow \alpha = 0$ *ou* $u = 0_E$.

2 Vectorial subspaces

Définition 2.1 *A non-empty subset F of a $\mathbb{K}$-vector space E, equipped with the rules + and ·, is a vector subspace of E. We denote it as F v.s. of E if and only if:*

1. $F \neq \emptyset$.
2. $\forall u, v \in F, \forall \alpha, \beta \in \mathbb{K} : \alpha u + \beta v \in F$.

Exemple 2.2 *Let's show that*

$$E_1 = \{(x, y, z) \in \mathbb{R}^3 / x + y - z = 0\}$$

is a vectorial subspace of $\mathbb{R}^3$.

1. $E_1 \neq \emptyset$, *because* $(0, 0, 0) \in E_1$.
2. $\forall u = (x, y, z) \in E_1, \forall v = (x', y', z') \in E_1, \forall \lambda, \mu \in \mathbb{R}$:

   $$\lambda u + \mu v = (\lambda x + \mu x', \lambda y + \mu y', \lambda z + \mu z').$$

   Or,

   $$(\lambda x + \mu x') + (\lambda y + \mu y') - (\lambda z + \mu z') = \lambda \underbrace{(x + y - z)}_{=0} + \mu \underbrace{(x' + y' - z)}_{=0} = 0.$$

   because $u = (x, y, z) \in E_1$ *et* $v = (x', y', z') \in E_1 \Rightarrow \lambda u + \mu v \in E_1$.

Conclusion: *E_1 is a vectorial subspace of $\mathbb{R}^3$.*

Proposition 2.3 *Every vectorial subspace of a $\mathbb{K}$-vector space E contains 0_E.*

3 Linearly independent family, linearly dependent family

Définition 3.1 *n non-zero vectors $v_1, v_2, ..., v_n$ in E are linearly independent (free) if and only if*

$$\forall a_1, a_2, ..., a_n \in \mathbb{K} : (a_1v_1 + a_2v_2 + ... + a_nv_n = 0) \Rightarrow (a_1 = a_2 = ... = a_n = 0).$$

Exemple 3.2 *Consider the vectors in $\mathbb{R}^3$: $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$. Let's show that they are linearly independent:*

$$a_1e_1 + a_2e_2 + a_3e_3 = a_1(1, 0, 0) + a_2(0, 1, 0) + a_3(0, 0, 1)$$
$$= (0, 0, 0) \implies \begin{cases} a_1 = 0 \\ a_2 = 0 \\ a_3 = 0 \end{cases}$$

Thus, the family e_1, e_2, e_3 is linearly independent.

Définition 3.3 *n vectors $v_1, v_2, ..., v_n$ in E are linearly dependent if and only if the equation*

$$a_1v_1 + a_2v_2 + ... + a_nv_n = 0,$$

has at least one solution with $a_i \neq 0$.

Exemple 3.4 *Consider the vectors in $\mathbb{R}^3$: $u_1 = (1, 1, 0)$, $u_2 = (0, 1, 1)$, and $u_3 = (1, 2, 1)$. Let's show that they are linearly dependent:*

$$a_1u_1 + a_2u_2 + a_3u_3 = a_1(1, 1, 0) + a_2(0, 1, 1) + a_3(1, 2, 1)$$
$$= (0, 0, 0)$$

$$\Rightarrow \begin{cases} a_1 + a_3 & = 0 \\ a_1 + a_2 + 2a_3 & = 0 \\ a_2 + a_3 & = 0 \end{cases}$$

$$\Rightarrow \begin{cases} a_1 & = -a_3 \\ a_2 & = -a_3 \end{cases}$$

So, the family u_1, u_2, u_3 is linearly dependent (It can be noticed that $u_3 = u_1+u_2$, which implies that the family u_1, u_2, u_3 is linearly dependent).

Définition 3.5 *Let E be a $\mathbb{K}$-vector space, and $v_1, v_2, ..., v_n$, n vectors in E. Any expression of the form*

$$\alpha_1v_1 + \alpha_2v_2 + ... + \alpha_nv_n,$$

where $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{K}$ are the coefficients, is called a linear combination of the vectors $v_1, v_2, ..., v_n$.

4 Bases and dimension of a vectorial space

Définition 4.1 *A family $v_1, v_2, ..., v_n$ is said to be generating for E if and only if*

$$\forall v \in E, \exists \alpha_i \in \mathbb{K} \text{ such that } v = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n,$$

that is:

$$E = \{\sum_{i=1}^{n} \alpha_i v_i / \alpha_i \in \mathbb{K}\},$$

and we haveécrit, $E = lin\{v_1, v_2, ..., v_n\}$ or $E = Vect\{v_1, v_2, ..., v_n\}$.

Exemple 4.2 *Let $E = \{(x+y, y-3x, x) \in \mathbb{R}^3 | x, y \in \mathbb{R}\}$.
For $u \in E \Rightarrow u = (x+y, y-3x, x) = (x, -3x, x) + (y, y, 0) = x(1, -3, 1) + y(1, 1, 0)$.
So {(1,-3,1),(1,1,0)} is a generating family for E. E.*

Définition 4.3 *A base of a vector space E is any family that is both linearly independent and generating for E.*

Exemple 4.4 *The family e_1, e_2, e_3 with $e_1 = (1, 0, 0)), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$ is a basis for $\mathbb{R}^3$. We have already shown that it is linearly independent. Let's now show that it is generating:
For $u \in \mathbb{R}^3$ we have: $u = (x, y, z) = (x, 0, 0) + (0, y, 0) + (0, 0, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)$.
Therefore, e_1, e_2, e_3 is a generating family for $\mathbb{R}^3$. Thus, it is a basis for $\mathbb{R}^3$ and is called the canonical basis of $\mathbb{R}^3$.*

Définition 4.5 *If $e_1, e_2, ..., e_n$ is a basis for the vector space E, then E is of finite dimension n. We denote it as $dimE = n$. By convention, $dim0_E = 0$.*

5 Linear application

Let E and F be two $\mathbb{K}$-vectorial spaces.

Définition 5.1 *A linear map T from E to F is defined as linear if and only if:*

$$\forall u, v \in E, \forall \alpha, \beta \in \mathbb{K} : T(\alpha u + \beta v) = \alpha T(u) + \beta T(v).$$

Exemple 5.2 *$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ a map defined by $T(x, y) = (x+y, x-y)$. T is linear. Indeed, let $u = (x, y), v = (x', y') \in \mathbb{R}^2$ and let $\alpha, \beta \in \mathbb{R}$:*

$$T(\alpha u + \beta v) = T(\alpha(x, y) + \beta(x', y')) = T(\alpha x + \beta x', \alpha y + \beta y').$$
$$= (\alpha x + \beta x' + \alpha y + \beta y', \alpha x + \beta x' - \alpha y - \beta y').$$
$$= (\alpha(x+y) + \beta(x'+y'), \alpha(x-y) + \beta(x'-y')).$$
$$= (\alpha(x+y), \alpha(x-y)) + (\beta(x'+y'), \beta(x'-y')).$$
$$= \alpha(x+y, x-y) + \beta(x'+y', x'-y') = \alpha T(x, y) + \beta T(x', y') = \alpha T(u) + \beta T(v).$$

Proposition 5.3 *If T from E to F is a linear map, then:*
1. $\forall u \in E : T(-u) = -T(u)$.
2. $T(0_E) = 0_F$.

Définition 5.4 *Let $T : E \longrightarrow F$ be a linear map. The kernel of T, denoted as $ker(T)$, is defined as*

$$kerT = \{u \in E;\ T(u) = 0_F\}.$$

Définition 5.5 *Let $T : E \longrightarrow F$ be a linear map. The image of T, denoted as $Im(T)$, is defined as*

$$ImT = \{v \in F;\ v = \ T(u)\ avec\ u \in E\} = T(E).$$

Proposition 5.6 *Let $T : E \longrightarrow F$ be a linear map. Then, $ker(T)$ is a subspace of E, and $Im(T)$ is a subspace of F.*

Theorem 5.7 *Let $T : E \longrightarrow F$ a linear application, then:*
1. T is injective $\Longleftrightarrow kerT = \{0_E\}$.
2. T is surjective $\Longleftrightarrow ImT = F$.

Theorem 5.8 (Kernel-Image Theorem) *Let $T : E \longrightarrow F$ be a linear map, with $dimE = n$ (finite), then*

$$dimE = dimkerT + dimImT.$$

Proposition 5.9 *If $T : E \longrightarrow F$ is linear and $dimE = dimF = n$, then the following properties are equivalent:*

- *T is bijective.*
- *T is injective.*
- *T is surjective.*

6 Rank of a linear map

Définition 6.1 (Rank of a linear map) *Let $T : E \longrightarrow F$ be a linear map. The rank of T is defined as the dimension of $Im(T)$. It is denoted as*

$$rg(T) = dim(Im(T)) = dim(T(E)).rg(T) = dim(Im(T)) = dim(T(E)).$$