Final Exam of Mathematics 2

Electronic devices are not allowed during the exam

Exercise1

Consider the following system:

$$(S) : \begin{cases} 3x - 2y + 4z = -7, \\ 5x + 7y - 3z = 16, \\ x + y - z = 6. \end{cases}$$

1) Write the matrix A associated with this system.
2) Calculate A^2.
3) Calculate det A by expanding the determinant of matrix A along the first row.
4) Calculate det A using the Sarrus rule.
5) Determine if Cramer's rule is applicable to solve the given system of equations. If yes, solve the system.

Exercise2

1. Calculate the following integral:

$$\int \frac{x}{x^2 + 2x - 3}\, dx$$

2. Deduce:

$$\int \frac{e^x}{e^x - 3e^{-x} + 2}\, dx$$

Exercise3

Consider the differential equation (E) $y' + y = xe^{-x}$

1. Find the solution of the homogeneous equation $y' + y = 0$
2. Look for a particular solution of the form $y_0(x) = k(x)e^{-x}$
3. Deduce the general solution of E.

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Exercise4

To solve the following differential equation:

$$y'' - 2y' + y = 0.$$

Good luck!

SolutionExercice 1

Solution.

$$(S):\begin{cases}3x-2y+4z=-7,\\5x+7y-3z=16,\\x+y-z=6.\end{cases}$$

1) $A=\begin{pmatrix}3&-2&4\\5&7&-3\\1&1&-1\end{pmatrix}$.

2) $A^2=\begin{pmatrix}3&-2&4\\5&7&-3\\1&1&-1\end{pmatrix}\begin{pmatrix}3&-2&4\\5&7&-3\\1&1&-1\end{pmatrix}=\begin{pmatrix}3&-16&14\\47&36&2\\7&4&2\end{pmatrix}$.

3) $det\ A=\begin{vmatrix}3&-2&4\\5&7&-3\\1&1&-1\end{vmatrix}=3\begin{vmatrix}7&-3\\1&-1\end{vmatrix}+2\begin{vmatrix}5&-3\\1&-1\end{vmatrix}+4\begin{vmatrix}5&7\\1&1\end{vmatrix}=24.$

4)$det\ A=\left|\begin{array}{ccc|cc}3&-2&4&3&-2\\5&7&-3&5&7\\1&1&-1&1&1\end{array}\right|=3\times 7(-1)+(-2)\times(-3)\times 1+4\times 5\times 1-4\times 7\times 1-3\times(-3)\times 1-(-2)\times 5\times(-1)=-24.$

5)Comme $det\ A=-24\neq 0$, alors: la méthode de Cramer est applicable et le système admet une unique solution:

$$x=\frac{det\ A_x}{det\ A}=\frac{\begin{vmatrix}-7&-2&4\\16&7&-3\\6&1&-1\end{vmatrix}}{-24}=\frac{-72}{-24}=3,$$

$$y=\frac{det\ A_y}{det\ A}=\frac{\begin{vmatrix}3&7&4\\5&16&-3\\1&6&-1\end{vmatrix}}{-24}=\frac{48}{-24}=-2,$$

$$z=\frac{det\ A_z}{det\ A}=\frac{\begin{vmatrix}3&-2&-7\\5&7&16\\1&1&6\end{vmatrix}}{-24}=\frac{120}{-24}=-5,$$

L'ensemble de solution: $S=\{(x,y,z)\}=\{(3,-2,-5)\}$.

Solution.

$$(S):\begin{cases}3x-2y+4z=-7,\\5x+7y-3z=16,\\x+y-z=6.\end{cases}$$

1) $A=\begin{pmatrix}3&-2&4\\5&7&-3\\1&1&-1\end{pmatrix}$. ....**[2 points]**

2) $A^2=\begin{pmatrix}3&-2&4\\5&7&-3\\1&1&-1\end{pmatrix}\begin{pmatrix}3&-2&4\\5&7&-3\\1&1&-1\end{pmatrix}=\begin{pmatrix}3&-16&14\\47&36&2\\7&4&2\end{pmatrix}$. **[2 points]**

3) $det\ A=\begin{vmatrix}3&-2&4\\5&7&-3\\1&1&-1\end{vmatrix}=3\begin{vmatrix}7&-3\\1&-1\end{vmatrix}+2\begin{vmatrix}5&-3\\1&-1\end{vmatrix}+4\begin{vmatrix}5&7\\1&1\end{vmatrix}=24.$ **[2 points]**

4) $det\ A = \begin{vmatrix} 3 & -2 & 4 \\ 5 & 7 & -3 \\ 1 & 1 & -1 \end{vmatrix} \begin{matrix} 3 & -2 \\ 5 & 7 \\ 1 & 1 \end{matrix} = 3\times 7\times(-1) + (-2)\times(-3)\times 1 + 4\times 5\times 1 - 4\times 7\times 1 - 3\times(-3)\times 1 - (-2)\times 5\times(-1) = -24$. **[2 points]**

5) Comme $det\ A = -24 \neq 0$, alors la méthode de Cramer est applicable et le système admet une unique solution:

$$x = \frac{det\ A_x}{det\ A} = \frac{\begin{vmatrix} -7 & -2 & 4 \\ 16 & 7 & -3 \\ 6 & 1 & -1 \end{vmatrix}}{-24} = \frac{-72}{-24} = 3, \textbf{ [2 points]}$$

$$y = \frac{det\ A_y}{det\ A} = \frac{\begin{vmatrix} 3 & 7 & 4 \\ 5 & 16 & -3 \\ 1 & 6 & -1 \end{vmatrix}}{-24} = \frac{48}{-24} = -2, \textbf{ [2 points]}$$

$$z = \frac{det\ A_z}{det\ A} = \frac{\begin{vmatrix} 3 & -2 & -7 \\ 5 & 7 & 16 \\ 1 & 1 & 6 \end{vmatrix}}{-24} = \frac{120}{-24} = -5, \textbf{ [2 points]}$$

L'ensemble de solution: $S = \{(x, y, z)\} = \{(3, -2, -5)\}$. **[2 points]**

SolutionExercice 2

1. Calculer l'intégrale suivante:

$$\int \frac{x}{x^2 + 2x - 3}\, dx$$

   On a: $x^2 + 2x - 3 = (x-1)(x+3)$

   $\dfrac{x}{x^2+2x-3} = \dfrac{\alpha}{x-1} + \dfrac{\beta}{x+3} = \dfrac{1}{4(x-1)} + \dfrac{3}{4(x+3)}$

$$\begin{aligned} \int \frac{x}{x^2+2x-3}\, dx &= \int \left(dfrac14(x-1) + \frac{3}{4(x+3)} \right) dx \\ &= \frac{1}{4} \ln|x-1| + \frac{3}{4} \ln|x+3| + C, \quad C \in \mathbb{R}. \end{aligned}$$

2. En déduire:

$$\int \frac{e^x}{e^x - 3e^{-x} + 2}\, dx$$

   Posons $t = e^x \updownarrow dt = e^x dx$

$$\int \frac{e^x}{e^x - 3e^{-x} + 2}\, dx = \int \frac{t}{t - 3\frac{1}{t} + 2}\, dt = \int \frac{e^x}{e^x - 3e^{-x} + 2}\, dx = \int \frac{t}{t^2 + 2t - 3}\, dt$$

   d'après la question précédente:

$$\int \frac{e^x}{e^x - 3e^{-x} + 2}\, dx = \frac{1}{4} \ln|e^x - 1| + \frac{3}{4} \ln|e^x + 3| + C, \quad C \in \mathbb{R}$$

SolutionExercice 3

1. La solution de l'équation homogène, $y' + y = 0$ est:

$$\begin{aligned}
y' + y = 0 &\Longrightarrow \frac{dy}{dx} = -y \\
&\Longrightarrow \frac{dy}{y} = -dx \\
&\Longrightarrow \int \frac{1}{y}\, dy = -\int\, dx \\
&\Longrightarrow ln|y| = -x + c, /c \in \mathbb{R} \\
&\Longrightarrow e^{ln|y|} = e^{-x+c}, /c \in \mathbb{R} \\
&\Longrightarrow y = e^c e^{-x}, /c \in \mathbb{R} \\
&\Longrightarrow y_h = Ke^{-x}, /K \in \mathbb{R}
\end{aligned}$$

2.

3. Une solution particulière de l'équation $E2$, sous la forme $y_p(x) = P(x)e^{-x}$

$$\begin{aligned}
y' + y = xe^{-x} &\Rightarrow P'(x)e^{-x} - P(x)e^{-x} + P(x)e^{-x} = xe^{-x} \\
&\Rightarrow P'(x)e^{-x} = xe^{-x} \\
&\Rightarrow P'(x) = x \\
&\Rightarrow P(x) = \frac{x^2}{2} \\
&\Rightarrow y_p(x) = \frac{x^2}{2} e^{-x}.
\end{aligned}$$

4.

5. Alors la solution générale de l'équation $E1$ est:

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{x^2}{2} + K\right) e^{-x}, /K \in \mathbb{R}
\end{aligned}$$

SolutionExercice 4

Résoudre l'équation différentielle suivante:

$$y'' - 2y' + y = 0.$$

L'équation caractéristique de l'équation différentielle est $r^2 - 2r + 1 = 0$, qui admet 1 comme racine double. Ainsi, les solutions de l'équation différentielle sont:

$$y(t) = (K_1 + K_2 t)\, e^t, \ \ K_1, K_2 \in \mathbb{R}.$$

Model Answer for the Final Exam in MATH2

Exercise 1 (7.5 pts) (S) : $\begin{cases} 3x-2y+4z=-7 \\ 5x+7y-3z=16 \\ x+y-z=6 \end{cases}$

1) $A=\begin{pmatrix} 3 & -2 & 4 \\ 5 & 7 & -3 \\ 1 & 1 & -1 \end{pmatrix}$ 0.75 pt

2) $A^2=\begin{pmatrix} 3 & -2 & 4 \\ 5 & 7 & -3 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & -2 & 4 \\ 5 & 7 & -3 \\ 1 & 1 & -1 \end{pmatrix}=\begin{pmatrix} 3 & -16 & 14 \\ 47 & 36 & 2 \\ 7 & 4 & 2 \end{pmatrix}$ 2.25 pts

3) $\det(A)=\begin{vmatrix} 3 & -2 & 4 \\ 5 & 7 & -3 \\ 1 & 1 & -1 \end{vmatrix}=3\begin{vmatrix} 7 & -3 \\ 1 & -1 \end{vmatrix}+2\begin{vmatrix} 5 & -3 \\ 1 & -1 \end{vmatrix}+4\begin{vmatrix} 5 & 7 \\ 1 & 1 \end{vmatrix}=-24$ 1.5 pt

4) $\det(A)=\left|\begin{matrix} 3 & -2 & 4 \\ 5 & 7 & -3 \\ 1 & 1 & -1 \end{matrix}\right|\begin{matrix} 3 & -2 \\ 5 & 7 \\ 1 & 1 \end{matrix}\Bigg|=3.7.(-1)+(-2)(-3)(1)+4.5.1$
$-4.7.1-1.(-3)(3)-(-1)(5)(-2)=-24$ 1 pt

5) Since $\det(A)=-24\neq 0$, then Cramer's method is applicable and the system has the unique solution 0.5 pt

$$x=\frac{\det A_x}{\det A}=\frac{\begin{vmatrix} -7 & -2 & 4 \\ 16 & 7 & -3 \\ 6 & 1 & -1 \end{vmatrix}}{-24}=\frac{-72}{-24}=3$$ 0.5 pt

$$y=\frac{\det A_y}{\det A}=\frac{\begin{vmatrix} 3 & -7 & 4 \\ 5 & 16 & -3 \\ 1 & 6 & -1 \end{vmatrix}}{-24}=\frac{48}{-24}=-2.$$ 0.5 pt

$$z=\frac{\det A_z}{\det A}=\frac{\begin{vmatrix} 3 & -2 & -7 \\ 5 & 7 & 16 \\ 1 & 1 & 6 \end{vmatrix}}{-24}=\frac{120}{-24}=-5.$$ 0.5 pt

The solution set : $S=\{(3,-2,-5)\}$

Exercise 2: 6,5 pts

1. $\int \frac{x}{x^2+2x-3}\,dx$?

We have $(x^2+2x-3)=(x-1)(x+3)$ 0.5 pt

$$\frac{x}{x^2+2x-3}=\frac{\alpha}{x-1}+\frac{\beta}{x+3}=\frac{1}{4(x-1)}+\frac{3}{4(x+3)}$$ 1 pt

Then: $\int \frac{x}{x^2+2x+3}\,dx=\int \frac{1}{4(x-1)}\,dx+\int \frac{3}{4(x+3)}\,dx$

$$=\frac{1}{4}\ln|x-1|+\frac{3}{4}\ln|x+3|+C,\ C\in\mathbb{R}$$ 2 pts

2. Deduce: $\int \frac{e^x}{e^x-3e^{-x}+2}\,dx$?

We put: $t=e^x \Longrightarrow dt=e^x dx$ 1 pt

$$\int \frac{e^x}{e^x-3e^{-x}+2}\,dx=\int \frac{1}{t-3\frac{1}{t}+2}\,dt=\int \frac{t}{t^2+2t-3}\,dt.$$ 1 pt

According to the previous question:

$$\int \frac{e^x}{e^x-3e^{-x}+2}\,dx=\int \frac{t}{t^2+2t-3}\,dt$$

$$=\frac{1}{4}\ln|t-1|+\frac{3}{4}\ln|t+3|+C,\ C\in\mathbb{R}$$

$$=\frac{1}{4}\ln|e^x-1|+\frac{3}{4}\ln|e^x+3|+C,\ C\in\mathbb{R}$$ 1 pt

Exercise 3 : 4pts $y' + y = x e^{-x}$: (E)

1) The solution to the homogeneous equation is :

$y' + y = 0 \Longrightarrow y' = -y \Longrightarrow \frac{dy}{dx} = -y \Longrightarrow \frac{dy}{y} = -dx$

$\Longrightarrow \int \frac{1}{y} dy = \int -dx$ 0,5pt

$\Longrightarrow \ln|y| = -x + c, \quad c \in \mathbb{R}$ 0,5pt

$\Longrightarrow y_h = K e^{-x}, \quad K \in \mathbb{R}$ 0,5pt

2) Let's look for a particular solution of the form :

$y_p(x) = K(x) e^{-x}$

$y' + y = x e^{-x} \Longrightarrow K'(x) e^{-x} - K(x) e^{-x} + K(x) e^{-x} = x e^{-x}$ 0,5pt

$\Longrightarrow K'(x) e^{-x} = x e^{-x}$

$\Longrightarrow K'(x) = x$ 0,5pt

$\Longrightarrow K(x) = \frac{x^2}{2}$ 0,5pt

$\Longrightarrow y_p(x) = \frac{x^2}{2} e^{-x}$ 0,5pt

3) So, the general solution to equation (E) is :

$y = y_h + y_p$

$= \left(\frac{x^2}{2} + k\right) e^{-x}, \quad K \in \mathbb{R}$ 0,5pt

Exercise 4 : 2pts

$$y'' - 2y' + y = 0$$

The corresponding characteristic equation is:

$$r^2 - 2r + 1 = 0$$ 0.5pts

We have :

$\Delta = 0$ 0.5pts

then :

$r_1 = r_2 = 1 = \alpha$ 0.5pts

So, $y_h = (C_1 x + C_2) e^{\alpha x}$

$= (C_1 x + C_2) e^{x}$, $C_1, C_2 \in \mathbb{R}$. 0.5pts