University of Tlemcen
Faculty of Sciences
First year LMD-SM

Final Exam of the second Semester

Module: Mathematics 2 — Responsible: A. Rimouche
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Exercise 1 (07 pts)**.**

1. Find the real parameters a and b such that : $\forall x \in \mathbb{R}, \quad x^2 - 2x + 2 = (x+a)^2 + b$.
2. Determine all the primitives of the function defined by: $\forall x \in \mathbb{R},\ f(x) = \dfrac{1}{x^2 - 2x + 2}$.
3. Solve the following differential equation: $xu' = u^2 - 2u + 2$.
4. Deduce the solution of the following system:

$$\begin{cases} x^2 y' = 2x^2 - xy + y^2 \\ y(1) = 1. \end{cases}$$

Exercise 2 (05 pts)**.** Let n be a natural number. Knowing that for all t in a neighborhood of 0 and for all $\alpha \in \mathbb{Q}, \quad (1+t)^\alpha = 1 + \alpha t + \frac{\alpha(\alpha-1)}{2} t^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-(n-1))}{n!} t^n + o\left(t^n\right)$:

1. Determine the asymptotic expansion of order two in a neighborhood of 0 of $\sqrt{1+x^2}$.
2. Deduce $\lim\limits_{x \to 0} \dfrac{1 - \sqrt{1+x^2}}{x^2 - 1 + \sqrt{1+x^2}}$

Exercise 3 (08 pts)**.**

I. Let A be the matrix defined by

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 4 \\ -1 & -1 & -4 \end{pmatrix}.$$

1. Calculate the comatrix of A.
2. Is A an invertible matrix. If yes, determine its inverse A^{-1}.

II. For all $m \in \mathbb{R}$, we consider the following system

$$\begin{cases} x_1 + 2x_3 & = 4 \\ 2x_1 + mx_2 + 4x_3 & = 8 - m \\ -x_1 - mx_2 + \left(m^2 - 3m - 2\right) x_3 & = 4 \end{cases} \qquad (1)$$

1. Write the matrix form of the system (1).
2. For which values of the parameter m the system (1) is Cramerian.
3. Solve the system (1) when $m = 1$.

Good luck.

UNIVERSITY OF TLEMCEN
FACULTY OF SCIENCES
FIRST YEAR LMD-SM

THE MODEL ANSWER FOR THE FINAL EXAM OF THE SECOND SEMESTER

Module: MATHEMATICS 2 Responsible: A. RIMOUCHE

■ Exercise 1 (07 pts).

1. (01 pts) Find the real parameters a and b such that : $\forall\, x \in \mathbb{R}, \quad x^2 - 2x + 2 = (x+a)^2 + b$.

$\forall\, x \in \mathbb{R}$,

$$x^2 - 2x + 2 = x^2 - 2x + 1 - 1 + 2$$
$$= (x-1)^2 + 1. \qquad \boxed{00.50 \text{ pts}}$$

Then, by identification, we conclude that $a = -1$ and $b = 1$. $\boxed{02 \times 00.25 \text{ pts}}$

2. (01.50 pts) Determine all the primitives of the function defined by: $\forall\, x \in \mathbb{R}, \ f(x) = \dfrac{1}{x^2 - 2x + 2}$.

$\forall\, x \in \operatorname{dom} f$,

$$\int f(x)\,\mathrm{d}x = \int \frac{\mathrm{d}x}{x^2 - 2x + 2}$$
$$= \int \frac{\mathrm{d}x}{(x-1)^2 + 1} \qquad \boxed{00.50 \text{ pts}}$$
$$= \left\langle \begin{array}{c} u = x - 1, \\ \mathrm{d}u = \mathrm{d}x \end{array} \right\rangle$$
$$= \int \frac{\mathrm{d}u}{u^2 + 1}. \qquad \boxed{00.50 \text{ pts}}$$

Hence, the primitives of f are the functions F, s.t., $\forall\, x \in \operatorname{dom} f, \quad F(x) = \arctan(x-1) + C$ where $C \in \mathbb{R}$ is an integration constant. $\boxed{00.50 \text{ pts}}$

3. (02 pts) Solve the following differential equation: $xu' = u^2 - 2u + 2$.

$\forall\, x \in \operatorname{dom} u$,

$$xu' = u^2 - 2u + 2 \iff \frac{u'}{u^2 - 2u + 2} = \frac{1}{x}, \qquad \boxed{00.50 \text{ pts}}$$

then, the equation above is a separable variable equation, and we have

$$\int \frac{\mathrm{d}u}{u^2 - 2u + 2} = \int \frac{\mathrm{d}x}{x}. \qquad \boxed{00.50 \text{ pts}}$$

Using the second question we obtain that

$$\arctan(u-1) = \ln|x| + \mathrm{C}^{\mathrm{te}}. \qquad \boxed{00.50 \text{ pts}}$$

Consequently, the general solution of the equation $xu' = u^2 - 2u + 2$ is

$$u = \tan\left(\ln|x| + \mathrm{C}^{\mathrm{te}}\right) + 1. \qquad \boxed{00.50 \text{ pts}}$$

4. (02.50 pts) Deduce the solution of the following system:

$$\begin{cases} x^2y' = 2x^2 - xy + y^2 \\ y(1) = 1. \end{cases} \tag{S}$$

Begin by solving the equation

$$x^2y' = 2x^2 - xy + y^2. \tag{E}$$

$\forall\, x \in \operatorname{dom} y$,

$$\begin{aligned} (\text{E}) \iff y' &= \frac{2x^2 - xy + y^2}{x^2} \\ &= 2 - \frac{y}{x} + \left(\frac{y}{x}\right)^2. \end{aligned} \tag{E1}$$

Thus, (E) is a homogeneous equation. 00.50 pts

Using the substitution $u = y/x$, $y' = xu' + u$ in the equation (E1), we get 00.50 pts

$$xu' = 2 - 2u + u^2.$$ 00.50 pts

From the last question, the general solution of the equation above is

$$u = \tan\left(\ln|x| + \mathrm{C^{te}}\right) + 1.$$

Replacing the value of u in the equation above gives

$$\frac{y}{x} = \tan\left(\ln|x| + \mathrm{C^{te}}\right) + 1,$$

then, the general solution of the equation (E) is

$$y = x\tan\left(\ln|x| + \mathrm{C^{te}}\right) + x.$$ 00.50 pts

Since $y(1) = 1$, and by assigning the value of 1 to x the expression above, we get

$$\begin{aligned} y(1) &= \tan\left(\ln|1| + \mathrm{C^{te}}\right) + 1 \\ &= \tan \mathrm{C^{te}} + 1, \end{aligned}$$

then, $\mathrm{C^{te}} = 0$. Hence, the solution of the system (S) is $y = x\tan(\ln|x|) + x$. 00.50 pts

■ **Exercise 2** (05 pts). Let n be a natural number. Knowing that for all t in a neighborhood of 0 and for all $\alpha \in \mathbb{Q}$, $(1+t)^\alpha = 1 + \alpha t + \frac{\alpha(\alpha-1)}{2}t^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-(n-1))}{n!}t^n + o(t^n)$:

1. (02 pts) Determine the asymptotic expansion of order two in a neighborhood of 0 of $\sqrt{1+x^2}$.

For all x in a neighborhood of 0, we have

$$\sqrt{1+x^2} = 1 + \frac{x^2}{2} + o\left(x^2\right)$$ 02 × 01.00 pts

2. (03 pts) Deduce $\lim_{x\to 0} \frac{1-\sqrt{1+x^2}}{x^2-1+\sqrt{1+x^2}}$

$$\lim_{x\to 0} \frac{1-\sqrt{1+x^2}}{x^2-1+\sqrt{1+x^2}} = \lim_{x\to 0} \frac{1-\left(1+\frac{x^2}{2}+o\left(x^2\right)\right)}{x^2-1+\left(1+\frac{x^2}{2}+o\left(x^2\right)\right)}$$ 01.00 pts

$$= \lim_{x\to 0} \frac{-\frac{x^2}{2}+o\left(x^2\right)}{\frac{3x^2}{2}+o\left(x^2\right)}$$

$$= \lim_{x\to 0} \frac{-\frac{1}{2}+o\left(1\right)}{\frac{3}{2}+o\left(1\right)}$$ 01.00 pts

$$= -\frac{1}{3}.$$ 01.00 pts

■ **Exercise 3** (08 pts).

I. (04 pts) Let A be the matrix defined by

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 4 \\ -1 & -1 & -4 \end{pmatrix}.$$

1. (01.50 pts) Calculate the comatrix of A.

$$\text{Com}\, A = \begin{pmatrix} 0 & 4 & -1 \\ -2 & -2 & 1 \\ -2 & 0 & 1 \end{pmatrix}$$ 03 × 00.50 pts

2. (02.50 pts) Is A an invertible matrix. If yes, determine its inverse A^{-1}.

- (01 pts) Is A an invertible matrix.

$$\det A = \begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & 4 \\ -1 & -1 & -4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ -1 & -1 & -4 \end{vmatrix} = -2.$$ 00.50 pts

Since $|A| \neq 0$, then A is invertible. 00.50 pts

- (01.50 pts) Determine its inverse A^{-1}.

$$A^{-1} = \frac{1}{|A|}(\text{Com } A)^T$$
$$= -\frac{1}{2}\begin{pmatrix} 0 & -2 & -2 \\ 4 & -2 & 0 \\ -1 & 1 & 1 \end{pmatrix}.$$ 03 × 00.50 pts

II. (04 pts) For all $m \in \mathbb{R}$, we consider the following system

$$\begin{cases} x_1 + 2x_3 & = 4 \\ 2x_1 + mx_2 + 4x_3 & = 8 - m \\ -x_1 - mx_2 + (m^2 - 3m - 2)\,x_3 & = 4 \end{cases} \qquad (1)$$

1. (01.50 pts) Write the matrix form of the system (1).

The matrix form of the system (1) is

$$\begin{pmatrix} 1 & 0 & 2 \\ 2 & m & 4 \\ -1 & -m & m^2 - 3m - 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 - m \\ 4 \end{pmatrix}.$$ 03 × 00.50 pts

2. (01.50 pts) For which values of the parameter m the system (1) is Cramerian.

$$\begin{vmatrix} 1 & 0 & 2 \\ 2 & m & 4 \\ -1 & -m & m^2 - 3m - 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 \\ 1 & 0 & m^2 - 3m + 2 \\ -1 & -m & m^2 - 3m - 2 \end{vmatrix} = m^2(m - 3).$$ 00.50 pts

The system (1) is Cramerian iff $m^2(m - 3) \neq 0$, i.e., $m \in \mathbb{R} \setminus \{0; 3\}$. 02 × 00.50 pts

3. (01 pts) Solve the system (1) when $m = 1$.

For $m = 1$ the system (1) is equivalent to

$$\begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 4 \\ -1 & -1 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ 4 \end{pmatrix}.$$

Since $m \in \mathbb{R} \setminus \{0; 3\}$, then the system above is Cramerian.
From the question I., the solution of the studying system is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 & -2 & -2 \\ 4 & -2 & 0 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 7 \\ 4 \end{pmatrix} = \begin{pmatrix} 11 \\ -1 \\ -7/2 \end{pmatrix}.$$ 04 × 00.25 pts