Real Functions of a Real Variable

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1 Generalities

1.1 Bounded Functions

Definition 1.1 *We call a real function of a real variable any function from* $\mathbb{R}$ *or a subset of* $\mathbb{R}$ *to* $\mathbb{R}$.

Definition 1.2 *Let* $f : X \subset \mathbb{R} \to \mathbb{R}$ *and* $E \subset X$.

1. f *is bounded above in* $E \Leftrightarrow \exists M \in \mathbb{R}, \forall x \in E : f(x) \leq M$.
2. f *is bounded below in* $E \Leftrightarrow \exists m \in \mathbb{R}, \forall x \in E : f(x) \geq m$.
3. f *is bounded in* $E \Leftrightarrow \exists M \in \mathbb{R}, \exists m \in \mathbb{R}, \forall x \in E : m \leq f(x) \leq M$.

1.2 Monotone Functions

Definition 1.3 *Let* $f : E \subset \mathbb{R} \to \mathbb{R}$.

1. f *is increasing in* E *if* $\forall x, y \in E : x \leq y \Rightarrow f(x) \leq f(y)$.
2. f *is strictly increasing in* E *if* $\forall x, y \in E : x < y \Rightarrow f(x) < f(y)$.
3. f *is decreasing in* E *if* $\forall x, y \in E : x \leq y \Rightarrow f(x) \geq f(y)$.
4. f *is strictly decreasing in* E *if* $\forall x, y \in E : x < y \Rightarrow f(x) > f(y)$.
5. f *is monotonic if it is increasing or decreasing.*
6. f *is strictly monotonic if it is strictly increasing or strictly decreasing.*

1.3 Periodic Functions

Definition 1.4 *Let* $f : \mathbb{R} \to \mathbb{R}$. *We say that* f *is periodic if:*

$$\exists T > 0, \forall x \in \mathbb{R} : f(x + T) = f(x).$$

1.4 Even-Odd Functions

Definition 1.5 *Let $f : I \to \mathbb{R}$.*

1. *We say that f is even if $\forall x \in I : f(-x) = f(x)$.*
2. *We say that f is odd if $\forall x \in I : f(-x) = -f(x)$.*

2 Algebraic Operations on Functions

Let $E \subset \mathbb{R}$ and let $f : E \to \mathbb{R}$ and $g : E \to \mathbb{R}$.

2.1 Sum

We define the sum of two functions f and g as $f + g$, denoted as:

$$f + g : E \to \mathbb{R}, x \mapsto (f + g)(x) = f(x) + g(x).$$

2.2 Scalar Product

Let $\lambda \in \mathbb{R}$. The function $\lambda \cdot f$ is defined as:

$$\lambda \cdot f : E \to \mathbb{R}, x \mapsto (\lambda \cdot f)(x) = \lambda \cdot f(x).$$

2.3 Product

We define the product of two functions f and g as $f \cdot g$, denoted as:

$$f \cdot g : E \to \mathbb{R}, x \mapsto (f \cdot g)(x) = f(x) \cdot g(x).$$

2.4 Quotient

If $\forall x \in E : g(x) \neq 0$, then $\frac{f}{g}$ is defined as:

$$\frac{f}{g} : E \to \mathbb{R}, x \mapsto \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}.$$

3 Limit of a Function at a Point

Let f be a function defined on an interval $I \subset \mathbb{R}$, and let $x_0 \in I$. If $f(x)$ approaches l as x approaches x_0, we denote this as:

$$\lim_{x \to x_0} f(x) = l.$$

3.1 Properties

Proposition 3.1 *If the limit of a function at a point exists, then it is unique, and*

$$\left(\lim_{x \to x_0} f(x) = l\right) \Leftrightarrow \left(\lim_{\substack{x \to x_0 \\ >}} f(x) = \lim_{\substack{x \to x_0 \\ <}} f(x) = l\right).$$

3.2 Properties

Theorem 3.2 *Let f and g be two given functions. Then,*

1. *If* $\left(\lim_{x\to x_0} f(x) = l_1 \text{ and } \lim_{x\to x_0} g(x) = l_2\right)$, *then* $\left(\lim_{x\to x_0} (f(x)+g(x)) = l_1 + l_2\right)$.
2. *If* $\left(\lim_{x\to x_0} f(x) = l_1 \text{ and } \lim_{x\to x_0} g(x) = l_2\right)$, *then* $\left(\lim_{x\to x_0} (f(x)\cdot g(x)) = l_1 \cdot l_2\right)$.
3. *If* $\left(\lim_{x\to x_0} f(x) = l_1 \text{ and } \lim_{x\to x_0} g(x) = l_2 \neq 0\right)$, *then* $\left(\lim_{x\to x_0} \left(\frac{f(x)}{g(x)}\right) = \frac{l_1}{l_2}\right)$.
4. *If* $\lim_{x\to x_0} f(x) = 0$ *and g is a bounded function, then* $\lim_{x\to x_0} f(x)g(x) = 0$.
5. *(Squeeze Theorem) Let $a \in \mathbb{R}$ or $a = +\infty$ or $a = -\infty$.*

   If $\lim_a f(x) = l$ *and* $\lim_a g(x) = l$ *and* $f \le h \le g$, *then* $\lim_a h(x) = l$.

6. *(Comparison Theorem)*

$$\text{If } \lim_{+\infty} f(x) = +\infty \text{ and } g \ge f, \text{ then } \lim_{+\infty} g(x) = +\infty.$$

$$\text{If } \lim_{+\infty} f(x) = -\infty \text{ and } g \le f, \text{ then } \lim_{+\infty} g(x) = -\infty.$$

4 Continuity

Definition 4.1 *Let f be a function defined at a point x_0.*

1. *We say that f is right-continuous at x_0 if* $\lim_{\substack{x\to x_0\\ >}} f(x) = f(x_0)$.
2. *We say that f is left-continuous at x_0 if* $\lim_{\substack{x\to x_0\\ <}} f(x) = f(x_0)$.
3. *We say that f is continuous at x_0 if*

$$\lim_{x\to x_0} f(x) = \lim_{\substack{x\to x_0\\ >}} f(x) = \lim_{\substack{x\to x_0\\ <}} f(x) = f(x_0).$$

4. *f is continuous on an interval $I \subset \mathbb{R}$ if it is continuous at every point in I.*

4.1 Operations on Continuous Functions

Let f and g be two functions defined on an interval I, and let a be a real number in I. If functions f and g are continuous at a, then:

1. $\lambda.f$ is continuous at a ($\lambda \in \mathbb{R}$).
2. $f + g$ is continuous at a (the same applies to subtraction).
3. $f \cdot g$ is continuous at a.

4. $\dfrac{f}{g}$ is continuous at a if $g(a) \neq 0$ and is undefined at a if $g(a) = 0$.

5. If a function g is continuous at point a and a function f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

4.2 Continuity Extension

Let I be an interval, and $x_0 \in I$. If f is a function defined on $I \setminus \{x_0\}$, and $\lim_{x \to x_0} f(x) = l$ exists, then the function g defined as

$$g(x) = \begin{cases} f(x) \text{ if } x \neq x_0 \\ l \text{ if } x = x_0 \end{cases}$$

is called the continuity extension of f at x_0. The function g is then continuous at x_0.

4.3 Intermediate Value Theorem

Theorem 4.2 *If f is a continuous function on an interval $[a, b]$ with $f(a) \cdot f(b) < 0$, then there exists $\alpha \in]a, b[$ such that $f(\alpha) = 0$. Furthermore, if f is strictly monotonic on $[a, b]$, then α is unique.*

4.4 Strictly Monotonic Continuous Function

Let I be an interval in $\mathbb{R}$, and let f be a function defined on I, continuous and strictly monotonic. In this case, the following properties hold:

1. f is a bijective function from I to $f(I)$.

2. The inverse function $f^{-1} : f(I) \to I$ is continuous and strictly monotonic, following the same nature as f (if f is strictly increasing, then f^{-1} is also, and if f is strictly decreasing, then f^{-1} is as well).

3. The graphs of f and f^{-1} are symmetric with respect to the first bisector $y = x$.

5 Differentiation

5.1 Definitions

Let $f : I \to \mathbb{R}$ be a function defined on an interval $I \subset \mathbb{R}$.

1. We say that f is differentiable at a point $x_0 \in I$ if and only if the following limit exists and is finite:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = l, \qquad (1)$$

and we denote in this case $l = f'(x_0)$, called the derivative of f at point x_0.

2. If we replace x with $x_0 + h$ in the limit (1), then as x approaches x_0, h approaches 0, and we obtain:

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0).$$

3. We say that f is differentiable over I if and only if f is differentiable at every point in I.

4. If f is differentiable over I, we can define a new function called the derivative, denoted as f', which, at each point x_0 in I, associates the derivative $f'(x_0)$.

5. Geometric interpretation: The tangent line to the curve representing f at a point $(x_0, f(x_0))$ has a slope equal to $f'(x_0)$ and can be represented by the equation:

$$y = f'(x_0)\,(x - x_0) + f(x_0).$$

6. $$f \text{ is differentiable at } x_0 \Leftrightarrow \lim_{\substack{x \to x_0 \\ >}} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{\substack{x \to x_0 \\ <}} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0).$$

7. $$f \text{ is differentiable at } x_0 \Rightarrow f \text{ is continuous at } x_0.$$

5.2 Derivative Rules

If f and g are two differentiable functions, then the following rules apply:

1. $(f + g)' = f' + g'$.
2. $(f \cdot g)' = f'g + fg'$.
3. $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$.
4. $(f^\alpha)' = \alpha f' f^{\alpha - 1}$.
5. We denote, when they exist, $f = f^{(0)}$, $f' = f^{(1)}, f'' = f^{(2)}$ ... $f^{(n)} = \left[f^{(n-1)}\right]'$, and $f^{(n)}$ is called the nth derivative of f.
6. If f is differentiable over I and g is differentiable over $f(I)$, then $(g \circ f)$ is differentiable over I, and we have the derivative rule:

$$(g \circ f)' = f' \cdot (g' \circ f)\,.$$

7. If f is strictly monotonic and differentiable over I, then its reciprocal function f^{-1} is differentiable over $f(I)$, and we have the derivative rule:

$$\left(f^{-1}\right)' = \frac{1}{f' \circ f^{-1}}.$$

5.3 Some Theorems

Theorem 5.1 ***Rolle's Theorem***
Let f be a function that is continuous on $[a, b]$, differentiable on $]a, b[$, and such that $f(a) = f(b)$, then:

$$\exists c \in \,]a, b[\,;\; f'(c) = 0.$$

Theorem 5.2 ***Mean Value Theorem***
Let f be a function that is continuous on $[a, b]$, and differentiable on $]a, b[$, then:

$$\exists c \in \,]a, b[\,;\; f(b) - f(a) = \; f'(c)\,(b - a)\,.$$

Proposition 5.3 *Let f be a function that is continuous on $[a, b]$, and differentiable on $]a, b[$, then f is increasing (or decreasing) if and only if its derivative f' is positive (or negative).*

Theorem 5.4 ***L'Hôpital's Rule***
Let f and g be two functions that are continuous on an interval $I \subset \mathbb{R}$, except possibly at the point $x_0 \in I$. If $f(x_0) = g(x_0) = 0$ and $g'(x) \neq 0$ for all $x \in I \setminus \{x_0\}$, and if $\lim\limits_{x \to x_0} \frac{f'(x)}{g'(x)} = l$, then:

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = l.$$

5.3.1 Elementary Reciprocal Functions

1. The function

$$f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$$
$$x \mapsto f(x) = \sin(x)$$

is strictly increasing, therefore bijective and has an inverse function denoted as arcsin. Hence,

$$\arcsin : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$
$$x \mapsto f(x) = \arcsin(x)$$

with the property

$$\forall x \in [-1, 1], \quad \left(y = \arcsin(x) \Longleftrightarrow x = \sin(y) \text{ and } y \in [-\frac{\pi}{2}, \frac{\pi}{2}]\right).$$

Also,

$$\arcsin(\sin(x)) = x; \quad \forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \text{ and } \sin(\arcsin(x)) = x; \quad \forall x \in [-1, 1].$$

2. The function

$$f : [0, \pi] \longrightarrow [-1, 1]$$
$$x \mapsto f(x) = \cos(x)$$

is strictly decreasing, therefore bijective and has an inverse function denoted as arccos. Hence,

$$\arccos : [-1, 1] \longrightarrow [0, \pi]$$
$$x \mapsto f(x) = \arccos(x)$$

with the property

$$\forall x \in [-1, 1], \quad (y = \arccos(x) \Longleftrightarrow x = \cos(y) \text{ and } y \in [0, \pi]).$$

Also,

$$\arccos(\cos(x)) = x; \quad \forall x \in [0, \pi] \, and \cos(\arccos(x)) = x; \quad \forall x \in [-1, 1].$$

3. The function

$$f : \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\longrightarrow \mathbb{R}$$
$$x \mapsto f(x) = \tan(x)$$

is strictly increasing, therefore bijective and has an inverse function denoted as $\arctan(x)$. Hence,

$$\arctan(x) : \mathbb{R} \longrightarrow \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$$
$$x \mapsto f(x) = \arctan(x)$$

with the property

$$\forall x \in \mathbb{R}, \quad \left(y = \arctan(x) \Longleftrightarrow x = \tan(y) and y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\right).$$

Also,

$$\arctan(\tan(x)) = x; \quad \forall x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\, and \tan(\arctan(x)) = x; \quad \forall x \in \mathbb{R}.$$