Logic and Methods of Mathematical Reasoning

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1 Introduction

1.1 Course Objective

This course aims to familiarize students with the fundamental concepts of mathematical logic and to develop their skills in mathematical reasoning.

1.2 Course Outline

1. Introduction to propositional logic
2. Logical connectors
3. Truth tables
4. Methods of reasoning
    (a) Direct reasoning
    (b) Reasoning by contraposition
    (c) Reasoning by contradiction
    (d) Reasoning by counterexample
    (e) Reasoning by induction
5. Applications in mathematics

2 Propositional Logic

2.1 Introduction to Propositional Logic

Propositional logic is a branch of mathematical logic that deals with propositions, logical operations, and the determination of the truth or falsity of complex propositions. This section presents the basic concepts of propositional logic and explains its importance in mathematics.

2.2 Propositions and Their Operations

Propositions are statements that can be true or false. This subsection examines simple and complex propositions, as well as logical operations such as negation, conjunction, disjunction, implication, and equivalence.

Example Consider the propositions P: "It is raining" and Q: "I take an umbrella." The proposition "If P, then Q" can be expressed as $P \Rightarrow Q$ using logical connectors.

3 Logical Connectors

3.1 Negation, Conjunction, Disjunction, Implication, Equivalence

In this section, we will explore the basic definitions of each logical connector and understand how they work in propositional logic.

3.1.1 Negation of a Proposition

The **negation** of a proposition P is a new proposition denoted $\overline{P}$, which is true when P is false, and false when P is true. In other words, it reverses the truth value of the original proposition.

Example: If P represents "It is raining," then $\overline{P}$ represents "It is not raining."

3.1.2 Conjunction ($\wedge$)

The **conjunction** of two propositions P and Q is a new proposition denoted $P \wedge Q$, which is true only when both propositions P and Q are true.

Example: If P represents "It is sunny" and Q represents "I am going to the park," then $P \wedge Q$ means "It is sunny, and I am going to the park."

3.1.3 Disjunction ($\vee$)

The **disjunction** of two propositions P and Q is a new proposition denoted $P \vee Q$, which is true if at least one of the two propositions P and Q is true.

Example: If P represents "It is sunny" and Q represents "It is raining," then $P \vee Q$ means "It is sunny or it is raining."

3.1.4 Implication ($\Rightarrow$)

The **implication** of two propositions P and Q is a new proposition denoted $P \Rightarrow Q$, which is true except when P is true and Q is false. It generally expresses a cause-and-effect relationship.

Example: If P represents "If the ground is wet" and Q represents "Then it has rained," then $P \Rightarrow Q$ means "If the ground is wet, then it has rained."

3.1.5 Equivalence ($\Leftrightarrow$)

The **equivalence** of two propositions P and Q is a new proposition denoted $P \Leftrightarrow Q$, which is true when both propositions have the same truth value (both true or both false). It expresses a bidirectional relationship.

Example: If P represents "It is raining" and Q represents "The ground is wet," then $P \Leftrightarrow Q$ means "It is raining if and only if the ground is wet."

4 Truth Tables

Construction of Truth Tables

Truth tables are essential tools for determining the truth value of complex propositions. In this section, we will construct truth tables for two given propositions.

Truth Table for the Proposition "$(P \wedge Q) \vee \overline{P}$" To construct the truth table for the proposition "$(P \wedge Q) \vee \neg P$", follow these steps:

1. Identify the basic propositions: P and Q.
2. List all possible combinations of truth values for P and Q.
3. Evaluate the truth value of each component of the proposition using the appropriate logical connectors.
4. Record the truth value of the overall proposition.

Here is the truth table for "$(P \wedge Q) \vee \overline{P}$":

P	Q	$\overline{P}$	$P \wedge Q$	$(P \wedge Q) \vee \overline{P}$
True	True	False	True	True
True	False	False	False	False
False	True	True	False	True
False	False	True	False	True

5 Methods of Reasoning

5.1 Direct Reasoning

Direct reasoning is a method of proof where one starts with true assumptions to deduce a conclusion. This section explains how to use direct reasoning to prove mathematical statements.

5.1.1 Example (Direct Reasoning)

Show that if n is an even integer, then n^2 is an even integer.

5.2 Reasoning by Contraposition

Reasoning by contraposition involves showing that the negation of the statement implies a negation.

5.2.1 Exercise (Reasoning by Contraposition)

Use contraposition to show that if n is an odd integer, then n^2 is an odd integer.

5.3 Reasoning by Contradiction

Reasoning by contradiction is a proof technique where one assumes that the negation of the statement leads to a contradiction.

5.4 Reasoning by Counterexample

Show that: If we have $x^3 + x^2 + x - 3 \neq 0$, then $x \neq 0$.

5.5 Reasoning by Induction

Reasoning by induction is a proof method used to establish statements for all natural numbers. This section explains the three steps of induction: Initialization, Inductive Hypothesis, Inductive Step.

6 Applications in Mathematics

6.1 Concrete Examples of Applications

This section provides concrete mathematical examples.

6.1.1 Example

Show that the sum of two odd integers is an even integer.

6.1.2 Exercise

Use induction to prove that $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$ for all positive integers n.