University of Tlemcen
Faculty of Sciences
First year LMD-SM

Final Exam of the First Semester

Module: Mathematics 1 | Responsible: A. Rimouche
Date: January 16, 2024 | Duration: 01h 30min

Unauthorized documents

Exercise 1 (04 pts)**.**

I. Let a be a reel number. Give an example of a neighborhood of a.

II. Let $f : \mathbb{R} \to \mathbb{R},\ x \mapsto f(x) = \dfrac{x}{x^2+1}$.

(a) Show that: $\forall x \in \mathbb{R}, \quad 0 < \dfrac{1}{x^2+1} \leq 1$.

(b) Deduce that: $\forall x \in \mathbb{R}, \quad \left|\dfrac{x}{x^2+1}\right| \leq |x|$.

(c) Using Cauchy's Criteria, show that f is continuous at 0.

Exercise 2 (05 pts)**.**

I. Let P and Q be two statements. Using the truth table, verify the following equivalence: $\neg(P \wedge Q) \iff \neg P \vee \neg Q$.

II. Let X the universal set and $A, B \subset X$. Show that: $(A \cup B)^c = A^c \cap B^c$.

Exercise 3 (05 pts)**.** Let $f : \mathscr{D}_f \subseteq \mathbb{R} \to \mathbb{R},\ x \mapsto f(x) = \dfrac{x}{(x-2)(x+2)}$.

1. Determine $\mathscr{D}_f$ the domain of definition of the function f.
2. Find the parity of f.

Exercise 4 (06 pts)**.** Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \dfrac{x}{\sqrt{1+x^2}}$.

1. Show that f is injective on $\mathbb{R}_+$.
2. Show that f is surjective from $\mathbb{R}_+$ to $[0;1)$.
3. Determine the sets $\mathscr{D}_1, \mathscr{D}_2 \subseteq \mathbb{R}$ such that, $f : \mathscr{D}_1 \to \mathscr{D}_2$ is bijective. In this case, find the expression of f^{-1}.

Good luck.

UNIVERSITY OF TLEMCEN
FACULTY OF SCIENCES
FIRST YEAR LMD-SM

THE MODEL ANSWER FOR THE FINAL EXAM OF THE FIRST SEMESTER

Module: MATHEMATICS 1 — Responsible: A. RIMOUCHE

Exercise 1 (04 pts).

I. (01.00 pt). Let a be a reel number. Give an example of a neighborhood of a.

A neighborhood $\mathscr{N}_a$ of a point $a \in \mathbb{R}$ can be any open interval centered at a. **01.00 pt**

II. (03.00 pts). Let $f : \mathbb{R} \to \mathbb{R},\ x \mapsto f(x) = \dfrac{x}{x^2+1}$.

(a) (01.00 pt). Show that: $\forall\, x \in \mathbb{R}, \quad 0 < \dfrac{1}{x^2+1} \leq 1.$

Let $x \in \mathbb{R}$,

$$\begin{aligned} x^2 \geq 0 &\implies x^2+1 \geq 1 \\ &\implies \frac{1}{x^2+1} \leq 1. \end{aligned}$$ **00.50 pts**

Furthermore,

$$\begin{aligned} x^2 \geq 0 &\implies x^2+1 > 0 \\ &\implies \frac{1}{x^2+1} > 0. \end{aligned}$$ **00.50 pts**

Q.E.D

(b) (00.50 pts). Deduce that: $\forall\, x \in \mathbb{R}, \quad \left|\dfrac{x}{x^2+1}\right| \leq |x|.$

$\forall\, x \in \mathbb{R}$, we have,

$$\begin{aligned} \left|\frac{x}{x^2+1}\right| &= \left|\frac{1}{x^2+1}\right| |x| \\ &= \frac{1}{x^2+1}|x| && \because \left(\frac{1}{x^2+1} > 0\right) \\ &\leq |x|\,. && \because \left(\frac{1}{x^2+1} \leq 1\right) \end{aligned}$$ **00.50 pts**

Q.E.D

(c) (01.50 pts). Using Cauchy's Criteria, show that f is continuous at 0.

Let $\varepsilon > 0$.

If $|x| < \varepsilon$, then $|f(x) - f(0)| = \left|\frac{x}{x^2+1}\right| \leq |x-0| < \varepsilon$. **00.50 pts**

You just have to choose $\delta_\varepsilon = \varepsilon$ to obtain, **00.50 pts**

$\forall\, \varepsilon > 0, \exists\, \delta_\varepsilon > 0 : (\forall\, x \in \mathscr{N}_0 : |x-0| < \delta_\varepsilon \implies |f(x) - f(0)| < \varepsilon)$, i.e.,

$\lim_{x\to 0} f(x) = 0 = f(0)$. Thus, f is continuous at 0. **00.50 pts**

Exercise 2 (05 pts).

I. (02.00 pts). Let P and Q be two statements.

Using the truth table, verify the following equivalence: $\neg(P \wedge Q) \iff \neg P \vee \neg Q$.

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$	$\neg(P \wedge Q) \iff (\neg P \vee \neg Q)$	
0	0	1	1	0	1	1	1	00.50 pts
0	1	1	0	0	1	1	1	00.50 pts
1	0	0	1	0	1	1	1	00.50 pts
1	1	0	0	1	0	0	1	00.50 pts

II. (03.00 pts). Let X the universal set and $A, B \subset X$. Show that: $(A \cup B)^c = A^c \cap B^c$.

Let $x \in X$,

$$\begin{aligned} x \in (A \cup B)^c &\iff x \notin A \cup B && \text{00.50 pts} \\ &\iff (x \notin A) \wedge (x \notin B) && \text{01.00 pt} \\ &\iff (x \in A^c) \wedge (x \in B^c) && \text{01.00 pt} \\ &\iff x \in A^c \cap B^c. && \text{00.50 pts} \end{aligned}$$

Q.E.D

Exercise 3 (05 pts). Let $f : \mathscr{D}_f \subseteq \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x) = \dfrac{x}{(x-2)(x+2)}$.

1. (01.50 pts). Determine $\mathscr{D}_f$ the domain of definition of the function f.

$$\begin{aligned} \mathscr{D}_f &\triangleq \{x \in \mathbb{R} : (x-2)(x+2) \neq 0\} && \text{00.50 pts} \\ &= \mathbb{R} \setminus \{x \in \mathbb{R} : (x-2)(x+2) = 0\} && \text{00.50 pts} \\ &= \mathbb{R} \setminus \{-2; 2\}. && \text{00.50 pts} \end{aligned}$$

Q.E.D

2. (03.50 pts). Find the parity of f.

- $\mathbb{R} \setminus \{-2; 2\}$ is symmetric with respect to the origin, which implies that if $x \in \mathbb{R} \setminus \{-2; 2\}$, then $-x \in \mathbb{R} \setminus \{-2; 2\}$. 00.50 pts
- Let $x \in \mathbb{R} \setminus \{-2; 2\}$;

$$\begin{aligned} f(-x) &= \frac{(-x)}{((-x)+2)\,((-x)-2)} && \text{02} \times \text{00.50 pts} \\ &= \frac{-x}{(-x+2)\,(-x-2)} \\ &= \frac{-x}{-(x+2)\,(-(x-2))} && \text{01.00 pt} \\ &\triangleq -f(x), && \text{00.50 pts} \end{aligned}$$

then f is an odd function. 00.50 pts

Exercise 4 (06 pts). Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \dfrac{x}{\sqrt{1+x^2}}$.

1. (02.50 pts). Show that f is injective on $\mathbb{R}_+$.

Let $x_1,\ x_2 \in \mathbb{R}_+$:

$$\begin{aligned}
f(x_1) = f(x_2) &\iff \frac{x_1}{\sqrt{1+x_1^2}} = \frac{x_2}{\sqrt{1+x_2^2}} && \boxed{01.00\text{ pt}}\\
&\iff x_1\sqrt{1+x_2^2} = x_2\sqrt{1+x_1^2}\\
&\iff x_1^2\left(1+x_2^2\right) = x_2^2\left(1+x_1^2\right) \quad \because\ (x_1,\ x_2 \in \mathbb{R}_+) && \boxed{00.50\text{ pts}}\\
&\iff x_1^2 + x_1^2x_2^2 = x_2^2 + x_2^2x_1^2\\
&\iff x_1^2 - x_2^2 = 0\\
&\iff (x_1 - x_2)(x_1 + x_2) = 0\\
&\iff (x_1 = x_2) \vee (x_1 = -x_2) && \boxed{00.50\text{ pts}}\\
&\implies x_1 = x_2. \quad \because\ (x_1,\ x_2 \in \mathbb{R}_+) && \boxed{00.50\text{ pts}}
\end{aligned}$$

Q.E.D

2. (02.00 pts). Show that f is surjective from $\mathbb{R}_+$ to $[0;1)$.

Let $y \in [0;1)$. Let find $x \in \mathbb{R}_+$ such that

$$\begin{aligned}
y = f(x) &\iff y = \frac{x}{\sqrt{1+x^2}} && \boxed{01.00\text{ pt}}\\
&\iff y\sqrt{1+x^2} = x\\
&\iff y^2\left(1+x^2\right) = x^2 \quad \because\ (x \in \mathbb{R}_+ \text{ and } y \in [0;1)) && \boxed{00.50\text{ pts}}\\
&\iff y^2 + y^2x^2 - x^2 = 0\\
&\iff y^2 + x^2\left(y^2 - 1\right) = 0\\
&\iff x^2 = \frac{-y^2}{y^2-1}\\
&\iff x^2 = \frac{y^2}{1-y^2}\\
&\iff x = \frac{y}{\sqrt{1-y^2}}. \quad \because\ (x \in \mathbb{R}_+ \text{ and } y \in [0;1)) && \boxed{00.50\text{ pts}}
\end{aligned}$$

Q.E.D

3. (01.50 pts). Determine the sets $\mathscr{D}_1, \mathscr{D}_2 \subseteq \mathbb{R}$ such that, $f : \mathscr{D}_1 \to \mathscr{D}_2$ is bijective. In this case, find the expression of f^{-1}.

- Determine the sets $\mathscr{D}_1, \mathscr{D}_2 \subseteq \mathbb{R}$ such that, $f : \mathscr{D}_1 \to \mathscr{D}_2$ is bijective.

  f is bijective if and only if f is injective and f is surjective, which leads us to take $\mathscr{D}_1 = \mathbb{R}_+$ and $\mathscr{D}_2 = (0;1]$. $\boxed{02 \times 00.50\text{ pts}}$

- The expression of f^{-1}.

$$\begin{aligned}
f^{-1}:\ [0;1) &\longrightarrow \mathbb{R}_+\\
x &\mapsto f^{-1}(x) = \frac{x}{\sqrt{1-x^2}}
\end{aligned} \qquad \boxed{00.50\text{ pts}}$$