Faculty of Sciences
University of Tlemcen

Year Academic 2023-2024

Mathematics1 Final Exam

Exercise1

Show by induction:

$$1 + 2^2 + 3^2 + ... + n^2 = \frac{1}{6}n(n+1)(2n+1) \ \forall n \in \mathbb{N}^*$$

Exercise2

We consider the mapping:

$$f: \quad [-1,1] \longrightarrow \mathbb{R}$$
$$x \mapsto f(x) = \frac{1}{1+x^2}$$

1. Calculate $f^{-1}(\{2\})$ et $f^{-1}(\{\frac{1}{2}\})$.
2. Study the injectivity and surjectivity of f.
3. Is f bijective?

Exercise3

Study continuity and differentiability on the domain of definition of the following function, then calculate $f^{'}(0)$.

$$f(x) = \begin{cases} \dfrac{e^x - 1}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 1. \end{cases}$$

Exercise4

By carrying out an expansion limited in 0, to a higher order to be determined, calculate the equation of tangent at 0 of the graph of the following function and indicate the relative position of the graph and its tangent:

$$f(x) = \sqrt{1+2x} - \sqrt{1+x^2}.$$

Exercise5

Let the set E defined by

$$E = \{(x,y,z) \in \mathbb{R}^3;\ x + y + 3z = 0\}$$

Is the set E vector subspace?

Good luck!

Model answer of the final exam for MATH1

Exercise 1 (1,5 pt)

For all $n \in \mathbb{N}^*$ we put:

$$P(n): \quad 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6} n(n+1)(2n+1)$$

. $P(1)$: $1^2 = \frac{1}{6} 1 (1+2)(2+1)$ is true. 0.5

. Let $n \in \mathbb{N}^*$, assuming that $P(n)$ is true, then:

$$1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 = \frac{1}{6} n(n+1)(2n+1) + (n+1)^2 \quad 0.5$$

$$= \frac{1}{6}(n+1)\left[n(2n+1) + 6(n+1)\right]$$

$$= \frac{1}{6}(n+1)(2n^2+7n+6)$$

$$= \frac{1}{6}(n+1)(n+2)(2n+3) \quad 0.5$$

This proves that $P(n+1)$ is true.

Conclusion: $P(n)$ is true $\forall n \in \mathbb{N}^*$.

Exercise 2: (6,5 pts)

We considere the mapping: $f : [-1, 1] \longrightarrow \mathbb{R}$

$$x \longmapsto f(x) = \frac{1}{1+x^2}$$

1). Let's calculate $f^{-1}(\{2\}) = ?$

$$x \in f^{-1}(\{2\}) \Longleftrightarrow x \in [-1, 1] \text{ and } f(x) = 2 \quad 0.5$$

$$\Longleftrightarrow x \in [-1, 1] \text{ and } \frac{1}{1+x^2} = 2 \quad 0.5$$

$$\Longleftrightarrow x \in [-1, 1] \text{ and } x^2 = -\frac{1}{2} \quad \text{impossible} \quad 0.5$$

Hence $\nexists x \in f^{-1}(\{2\}) \Longrightarrow \boxed{f^{-1}(\{2\}) = \emptyset}$ 0.5

• Let's calculate $f^{-1}(\{\frac{1}{2}\}) = ?$

$x \in f^{-1}(\{\frac{1}{2}\}) \Longleftrightarrow x \in [-1, 1]$ and $f(x) = \frac{1}{2}$ 0.5

$\Longleftrightarrow x \in [-1, 1]$ and $\frac{1}{1+x^2} = \frac{1}{2}$ 0.5

$\Longleftrightarrow x \in [-1, 1]$ and $x^2 - 1 = 0$

$\Longleftrightarrow x \in [-1, 1]$ and $x = \pm 1$

$\Longleftrightarrow x = +1$ or $x = -1$ 0.5

Hence $\boxed{f^{-1}(\{\frac{1}{2}\}) = \{-1, 1\}}$ 0.5

2) • Injectivity of f?

From the first question, we have : $f(1) = f(-1) = \frac{1}{2}$. 1 pt

So f is not injective.

• Surjectivity of f?

From the first question : $\nexists x \in [-1, 1] \,/\, f(x) = 2$ 1 pt

So f is not surjective.

3) f is ~~not~~ neither injective, nor surjective, so it's not bijective. 0.5

Exercise 3 : 5 pts

$$f(x) = \begin{cases} \frac{e^x - 1}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

→ f is defined on $\mathbb{R}$. 0.5

→ f is differentiable on $\mathbb{R}^*$ because it's the quotient of two differentiable functions, particularity on $\mathbb{R}^*$. 1 pt

It follows that f is continuous on $\mathbb{R}^*$. 0.5

→ let's study the continuity at $x=0$?

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{e^x - 1}{x} = 1 = f(0)$$ 1pt

So f is continuous at 0.

→ let's study the differentiability at $x=0$?

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{\frac{e^x-1}{x} - 1}{x} = \lim_{x\to 0} \frac{e^x-1-x}{x^2}$$ 0,5

$$= \lim_{x\to 0} \frac{(e^x-1-x)'}{(x^2)'} = \lim_{x\to 0} \frac{e^x-1}{2x}$$ 0,5

$$= \lim_{x\to 0} \frac{(e^x-1)'}{(2x)'} = \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}$$ 0,5

f is differentiable at 0 and $\boxed{f'(0) = \frac{1}{2}}$ 0,5

We have then:

$$f'(x) = \begin{cases} \dfrac{xe^x - e^x + 1}{x^2} & \text{for } x \neq 0 \\ \dfrac{1}{2} & \text{for } x = 0 \end{cases}$$

Exercise 4: 5,5pts

$$f(x) = \sqrt{1+2x} - \sqrt{1+x^2}$$

We perform a second-order Taylor expansion around 0:

$$(1+u)^\alpha = 1 + \alpha u + \frac{\alpha(\alpha-1)}{2}u^2 + o(u^2)$$

For $\alpha = \frac{1}{2}$ we have:

$$\sqrt{1+u} = 1 + \frac{1}{2}u - \frac{1}{8}u^2 + o(u^2)$$

Now we replace u by $2x$:

$$\sqrt{1+2x} = 1 + x - \frac{x^2}{2} + o(x^2)$$ 1pt

We replace u by x^2:

$$\sqrt{1+x^2} = 1 + \frac{x^2}{2} + o(x^2)$$ 1pt

We obtain thus the Taylor expansion of f around 0:

$$\boxed{f(x) = x - x^2 + o(x^2)}$$ 1pt

The equation of the tangent at 0 for f is: $\boxed{y = x}$ 1pt

To determine the relative position of the graph of f and the line $y = x$, we look the difference:

$$f(x) - x = -x^2 + o(x^2) \leq 0$$ 1pt

We conclude that <u>f is below its tangent at 0</u>. 0.5

<u>Exercise 5:</u> 1,5pt

$$E = \{(x, y, z) \in \mathbb{R}^3 ;\ x + y + 3z = 0\}$$

- $E \neq \emptyset$ (because $(0, 0, 0) \in E$) 0.5
- Let $X = (x, y, z)$ and $X' = (x', y', z')$ elements of E. Then,

$X + X' = (x + x', y + y', z + z')$ is also element of E. Indeed,

$$(x + x') + (y + y') + 3(z + z') = (x + y + 3z) + (x' + y' + 3z') = 0$$ 0.5

Similary, $\forall$ all $\lambda \in \mathbb{R}$, we have:

$\lambda X = (\lambda x, \lambda y, \lambda z)$ is an element of E, since

$$\lambda x + \lambda y + 3\lambda z = \lambda(x + y + 3z) = 0$$ 0.5

<u>E is thus a subspace of $\mathbb{R}^3$</u>