University of Tlemcen
Faculty of Sciences
First year LMD-SM
Mathematics 1

Tutorial Sheet number 02

Exercise 1. Write in an interval form the following sets, and determine when it exists the maximum, minimum, supremum and infemum

i) $A = \{x \in \mathbb{R} : (x \leq 2) \wedge (x + 3 > 0)\}$;

ii) $B = \{x \in \mathbb{R} : |x| - 1 \geq 0\}$;

iii) $C = \{x \in \mathbb{R} : \lfloor x \rfloor + 3 = 0\}$;

iv) **Optional.** $D = \{x \in \mathbb{R} : (-x + 7 \geq 0) \vee (x > \pi^2)\}$;

v) **Optional.** $E = \{x \in \mathbb{R} : x^2 + 2|x| - 3 \leq 0\}$;

vi) **Optional.** $F = \left\{x \in \mathbb{R} : \sqrt{x^2 - 2} \leq 2\right\}$.

Exercise 2. Optional. Let $n \in \mathbb{N}$. Prove that if $a \in \mathbb{N}$ is an n digit number, then $n = \lfloor \log a \rfloor + 1$. log is the logarithm to base 10.

Exercise 3. Optional. For each of the following problems, decide whether the solutions to the equation of the unknown $y \in \mathbb{R}$ constitute a rule of a function (map) of the variable $x \in \mathbb{R}$ or not:

(i) $x^4 = y^4$; (ii) $x^2 + y^2 = 0$; (iii) $x^2 + y^2 = 1, \quad y \geq 0$.

Exercise 4. Determine the domain of definition $\mathscr{D}_f \subseteq \mathbb{R}$ of the following real functions defined by

(1) $f_1(x) = \frac{\sqrt{x+1}}{x-4}$;

(2) $f_2(x) = \frac{1}{\sqrt{|x|-x}}$;

(3) $f_3(x) = \frac{\ln(x-1)}{\lfloor x \rfloor - 2x}$;

(4) $f_4(x) = \sqrt{\frac{-3}{x^2-5x+4}}$;

(5) **Optional.** $f_5(x) = \frac{14-x}{\ln(x^2-4)}$;

(6) **Optional.** $f_6(x) = \ln\left(\sqrt{\frac{3x-1}{x+4}}\right)$;

(7) **Optional.** $f_1 f_5(x)$;

(8) **Optional.** $f_4 \circ f_6(x)$;

(9) **Optional.** $\frac{f_2}{f_3}(x)$.

Exercise 5.

1. In each case, verify that T is the period of f.

   (1) $f : \mathbb{R} \to \mathbb{R}, x \mapsto f(x) = 4\cos\left(2\left(x + \frac{\pi}{4}\right)\right) - 3, \quad T = \pi$;

   (2) $f : \mathbb{R} \setminus \left\{\frac{5}{24} + \frac{k}{4} \mid k \in \mathbb{Z}\right\} \to \mathbb{R}, x \mapsto f(x) = \tan\left(4\pi^2 x - \frac{\pi}{3}\right), \quad T = \frac{1}{4\pi}$;

   (3) $f : \mathbb{R} \to \mathbb{R}, x \mapsto f(x) = \lfloor x \rfloor - x, \quad T = 1$;

2. **Optional.** Find the period of the following real functions of one real variable.

   (1) $f : x \mapsto f(x) = 3\sin(2x) - \tan(5x)$;

   (2) $f : x \mapsto f(x) = x - \lfloor x \rfloor + \cos(\pi(x+1))$.

Exercise 6. Find the parity of the following functions of one real variable. Justify your answer.

(1) $f : \mathscr{D}_f \to \mathbb{R},\ x \mapsto f(x) = \frac{|x|}{x}$;

(2) $f : \mathscr{D}_f \to \mathbb{R},\ x \mapsto f(x) = (x^2 + 4)(x - 2)(x + 2)$;

(3) $f : \mathscr{D}_f \to \mathbb{R},\ x \mapsto f(x) = \sqrt{|x|}$;

(4) $f : \mathscr{D}_f \to \mathbb{R},\ x \mapsto f(x) = x - \frac{1}{x}$;

(5) $f : \mathscr{D}_f \to \mathbb{R},\ x \mapsto f(x) = \lfloor x \rfloor$;

(6) **Optional.** $f : \mathscr{D}_f \to \mathbb{R},\ x \mapsto f(x) = \frac{1+x}{1+x} + \frac{1-x}{1+x}$;

(7) **Optional.** $f : \mathscr{D}_f \to \mathbb{R},\ x \mapsto f(x) = \frac{x^3-1}{x-1}$;

(8) **Optional.** $f : \mathscr{D}_f \to \mathbb{R},\ x \mapsto f(x) = \begin{cases} \frac{x+1}{x} & \text{if } x > 0, \\ \frac{x-1}{x} & \text{if } x < 0. \end{cases}$

Exercise 7. Optional. Using the periodicity and the parity calculate the following: *i)* $\sin\left(\frac{5\pi}{3}\right)$; *ii)* $\cos\left(\frac{11\pi}{6}\right)$; *iii)* $\tan\left(\frac{5\pi}{6}\right)$.

Exercise 8.

1. Find the monotonicity of the following functions:
   (a) $f_1 : [0;\infty) \to \mathbb{R},\ x \mapsto \frac{1}{x^2+1}$;
   (b) $f_2 : (-1;1) \to \mathbb{R},\ x \mapsto \sqrt{1-x^2}$;
   (c) $f_3 : (-\infty;0) \to \mathbb{R},\ x \mapsto e^{-x^2+x-1}$.
   (d) $f_4 : (-\infty,8) \to \mathbb{R},\ x \mapsto \ln(-x+8)$
2. Find the *lub* (supremum) and *glb* (infemum) of each of the following sets:
   (i) $= \left\{\frac{1}{x^2+1} \mid x \in (0;a]\right\}$, $a > 0$;
   (ii) $= \left\{\sqrt{1-x^2} \mid x \in [-1;0]\right\}$;
   (iii) **Optional.** $= \{\sin x \mid 0 \le x \le 5\pi/4\}$;
   (iv) **Optional.** $= \{x^2+1 \mid x \in [-5;4]\}$.

Exercise 9. Optional. Let $S \subset \mathbb{R}$ be a non-empty set that is bounded above and $a \in \mathbb{R}$, let aS be the set $\{ax \mid x \in S\}$.

1. Show that, if $a > 0$, then the set $a\,S$ is bounded above and $\sup(aS) = a \sup S$.
2. Show that, if $a < 0$, then the set $a\,S$ is bounded below and $\inf(aS) = a \sup S$.

Exercise 10. Optional. Let $A, B \subset \mathbb{R}$ be non-empty that are bounded from above, and define $A + B = \{x + y \mid (x \in A) \wedge (y \in B)\}$.

1. Show that $A + B$ is bounded above, and $\sup(A + B) = \sup A + \sup B$.
2. Deduce that if $c \in \mathbb{R}$ and $A + c$ is the set $\{x + c \mid x \in A\}$, then $A + c$ is bounded above and $\sup(A + c) = \sup A + c$.

Exercise 11. Decompose the following rational expressions into partial fractions

(a) $\frac{x}{(x+1)(x-2)}$;
(b) $\frac{2x}{(x^2+x+1)(x-2)}$;
(c) $\frac{2x^3+x^2-x+1}{x^2-2x+1}$;
(d) **Optional.** $\frac{x+3}{(x+1)^2(x-2)}$;
(e) **Optional.** $\frac{x^2}{x^2+1}$;
(f) **Optional.** $\frac{x^4+1}{(x+1)^2(x^2+1)}$.

Exercise 12.

1. Evaluate the following expressions, giving the answer in radians.
   (a) $\arcsin\left(\sqrt{2}/2\right)$;
   (b) $\arccos\left(-\sqrt{3}/2\right)$;
   (c) $\cos(\arctan(4))$;
   (d) $\cos(\arcsin(3/7))$;
   (e) **Optional.** $\arctan\left(\sqrt{3}\right)$;
   (f) **Optional.** $\arcsin(\cos(\pi/4))$.
2. **Optional.** For all $x \in \mathbb{R}$, evaluate the following expressions,
   (a) $\sin(\arctan(3x))$;
   (b) $\cos(\arctan(4x))$.

Exercise 13. Show that: $\forall\, x \in [-1;1]$, (1) $\cos(\arcsin x) = \sqrt{1-x^2}$; (2) $\sin(\arccos x) = \sqrt{1-x^2}$.