Tutorial Sheet number 01

Exercise 1. Let P, Q and R be three statements. Using the truth table, verify the following equivalences
(i) $\neg(P \implies Q) \iff P \wedge \neg Q$; (contradiction principle) (ii) $P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R)$;
(iii) $(P \implies Q) \iff (\neg Q \implies \neg P)$; (contraposition principle)
(iv) **Optional.** $\neg(P \wedge Q) \iff \neg P \vee \neg Q$; (v) **Optional.** $\neg(P \vee Q) \iff \neg P \wedge \neg Q$.
(vi) **Optional.** $\Big[(P \vee Q) \implies R\Big] \iff \Big[(P \implies R) \wedge (Q \implies R)\Big]$ (vii) **Optional.** $P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R)$;

Exercise 2. Suppose P is true, Q is true, R is false and S is true. Find the truth value of

(i) $(S \vee P) \wedge (Q \wedge \neg S)$;

(ii) $(S \vee P) \wedge (\neg R \vee \neg S)$;

(iii) **Optional.** $\neg\left[(S \wedge P) \vee \neg R\right]$;

(iv) **Optional.** $(\neg P \vee Q) \vee (S \wedge R)$.

Exercise 3.

1. Are the following statements *true* or *false*

   (a) $\exists x \in \mathbb{N} : \forall y \in \mathbb{R}, \quad x + y > 0$;

   (b) $\forall x \in \mathbb{R}, \exists y \in \mathbb{Z} : \quad x + y = 1$;

   (c) $\exists x \in \mathbb{D} : \exists y \in \mathbb{Q}, \quad xy = 0$;

   (d) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \quad x - y = y - x$.

2. Give their negation.

Exercise 4. Given that $U = \{-2, -1, 0, 1, 2, 3, 4, 5, 6, 7\}$, $X = \{\text{Even numbers in } U\}$, $Y = \{\text{Prime numbers in } U\}$ and $Z = \{-2, 3, 5, 6\}$,

1. Which ones are correct *i)* $2 \in U$; *ii)* $-2 \subset Z$; *iii)* $\{3\} \subset Y$; *iv)* $\emptyset \in U$; *v)* $\emptyset \subset U$; *vi)* $\{\emptyset\} \subset U$

2. Perform the following operations *i)* $X \cup Y$; *ii)* $(X \cap Y)^c$; *iii)* $X \cap (Y \setminus Z)$; *iv)* The cardinality of $X \cup Y$.

Exercise 5. Optional. Let $A = \{x \in \mathbb{Z} : x > 5\}$, $B = \{x \in \mathbb{Z} : x < 9\}$ and $C = \{x \in \mathbb{Z} : 5 \leq x \leq 7\}$. Find *i)* $A \cap B$; *ii)* $B \cup C$; *iii)* $(A \cup C)^c$; *iv)* $(A^c \cup B)^c$.

Exercise 6. Let X a set and $A, B \subset X$. Show that (a) $(A \cap B)^c = A^c \cup B^c$; (b) $A \setminus B = A \cap B^c$; (c) **Optional.** $(A \cup B)^c = A^c \cap B^c$.

Exercise 7. Optional. Let $A; B; C$ be three subsets of a set X. Show each of the following set equalities

1) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

2) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

3) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

4) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

Exercise 8. Let X a set and $A, B \subset X$. Prove by contradiction that, if $A \subset B$, then $A \setminus B = \emptyset$

Exercise 9. Prove by contraposition that

(i) $\forall n \in \mathbb{Z}, \quad n^3 + 5$ is odd $\implies$ n is even.

(ii) **Optional.** for all real numbers a and b, if ab is irrational, then a is irrational or b is irrational.

Exercise 10. Prove by mathematical induction

a) $\forall n \in \mathbb{N}, \sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$;

b) **Optional.** $\forall x \in \mathbb{R}_+, \forall n \in \mathbb{N}, (1+x)^n \geq 1 + nx$.

Exercise 11. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{x}{x^2+1}$. Is f injective ? surjective ? bijective ?

Exercise 12. Let $f : \mathbb{R} \setminus \{-1; 1\} \to \mathbb{R} \setminus \{0\}$ defined by $f(x) = \frac{x+1}{x-1}$, $g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $g(x) = 1/x$. Determine $g \circ f$.

Exercise 13. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 + 1$.

1. Find
   (a) $f([0; 2])$;
   (b) $f^{-1}((-1; 1))$;
   (c) $f^{-1}([2; 3])$.

2. If $g : \mathbb{R} \to \mathbb{R}$ is the map $g(x) = x^3$. Find

   (a)$(g \circ f)^{-1}([0; 2])$;(c) $(f \circ g)^{-1}([0; 2])$;

   (b) $(g \circ f)([0; 2])$; (d) $(f \circ g)([0; 2])$;

Exercise 14. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Prove that

1. if f and g are both injective, then $g \circ f$ is injective;
2. if f and g are both surjective, then $g \circ f$ is surjective.
3. if $g \circ f$ is surjective, then g is surjective.
4. if$g \circ f$ is surjective and g is injective, then f is surjective.

Exercise 15. Optional. Let $f : A \to B$ be a map. For subsets $K, L \subset A$ and $M, N \subset B$, the following hold

(a) $f(K \cup L) = f(K) \cup f(L)$.

(b) $f^{-1}(M \cup N) = f^{-1}(M) \cup f^{-1}(N)$.

(c) $f^{-1}(M \cap N) = f^{-1}(M) \cap f^{-1}(N)$.

(d) $f^{-1}(M \setminus N) = f^{-1}(M) \setminus f^{-1}(N)$.

(e) Show, by finding an explicit example, that in general, $f(K \setminus L) \neq f(K) \setminus f(L)$.

Exercise 16. Optional. Show that the function $f : A \to B$ is injective iff $f(K \cap L) = f(K) \cap f(L)$ for all subsets K, L of A.