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MATHEMATICS 1

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Part I

Set theory

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1. Sets and mappings

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1.1 Sets

Definition 1.1.1 Intuitively, a set A is a collection of objects. An object x containing in A is called *element* or *member*. We write

$$x \in A$$

to say *x is an element (member) of A, x belongs to A* or *A contains x*.

The number of elements of a set A is denoted by $\text{card}(A)$ (the cardinality of A).

A set is represented by

1. listing its members between a pair of braces. For example: $\{book, copybook, pen, pencil, paper, gum\}$ or $\{0, 1.2, \sqrt{2}, \pi, \frac{2}{3}, \ln 3, e^2\}$;
2. means of some defining property. For example: *the set of all letters of the English alphabet* or $\{x \mid x \text{ is an integer}\}$ also written by $\{x : x \text{ is an integer}\}$. In general, if P is a property, thus the set is often defined by $\{x : x \text{ has property } P\}$.

- x is not an element of the set X is symbolised by $x \notin A$ (x doesn't belong to A).
- The empty set is denoted by $\emptyset$ and is defined by for all x, x doesn't belong to $\emptyset$, i.e.,

$$\forall x, \quad x \notin \emptyset.$$

  Its cardinality $\operatorname{card}(\emptyset) = 0$.
- We call $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ a *digit*.
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$$\begin{aligned}
\mathbb{N} &= \text{the set of natural numbers,} \\
&\triangleq \{1; 2; 3; \ldots\} \\
&\triangleq [\![1; \infty[\![\\
\mathbb{Z} &= \text{the set of integers,} \\
&\triangleq \{\ldots; -3; -2; -1; 0; 1; 2; 3; \ldots\} \\
&\triangleq]\!]-\infty; \infty[\![\\
\mathbb{D} &= \text{the set of decimal numbers,} \\
&\triangleq \left\{ \frac{p}{2^n 5^m} : (p \in \mathbb{Z}) \wedge (n, m \in \mathbb{N}) \right\} \\
\mathbb{Q} &= \text{the set of rational numbers,} \\
&\triangleq \{p/q : (p \in \mathbb{Z}) \wedge (q \in \mathbb{N})\} \\
\mathbb{R} &= \text{the set of real numbers,} \\
&\triangleq \left\{ x : x = n + \sum_{k \in \mathbb{N}} \frac{a_k}{10^k} \text{ with } (\forall k \in \mathbb{N},\ a_k \text{ is a digit}) \wedge (n \in \mathbb{Z}) \right\} \\
&\triangleq (-\infty; \infty) \text{ or }]-\infty; \infty[\\
\mathbb{C} &= \text{the set of complex numbers,} \\
&\triangleq \{a + \mathrm{i}b : (a, b \in \mathbb{R}) \wedge (\mathrm{i}^2 = -1)\}.
\end{aligned}$$

- An *interval* is a subset of $\mathbb{R}$ defined by $\forall a, b \in \mathbb{R}$ ($a < b$),

$$\begin{aligned}
(a; b) &\triangleq \{x \in \mathbb{R} : a < x < b\}, && \text{Open interval;} \\
[a; b) &\triangleq \{x \in \mathbb{R} : a \leq x < b\}, && \text{Right-open interval;} \\
(a; b] &\triangleq \{x \in \mathbb{R} : a < x \leq b\}, && \text{Left-open interval;} \\
[a; b] &\triangleq \{x \in \mathbb{R} : a \leq x \leq b\}, && \text{Closed interval;} \\
(-\infty; b] &\triangleq \{x \in \mathbb{R} : x \leq b\}; \\
(-\infty; b) &\triangleq \{x \in \mathbb{R} : x < b\}; \\
[a; \infty) &\triangleq \{x \in \mathbb{R} : x \geq a\}; \\
(a; \infty) &\triangleq \{x \in \mathbb{R} : x > a\};
\end{aligned}$$

  we can replace '(' by '[' in the right-open interval and '(' by ']' in the left-open interval, thus, the open interval $(a; b)$ can be denoted by $]a; b[$

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- A *singleton* is a set consisting of exactly one member as in $A = \{b\}$. We have

$$x \in \{b\} \text{ if and only if (iff) } x = b.$$

Definition 1.1.2 Let A and B be two sets, we say that A is a *subset* of B or A is *contained* in B ($A \subseteq B$), iff, every member of A is a member of B (for all x, if $x \in A$, then $x \in B$), i.e.,

$$A \subseteq B \iff (\forall x, \quad x \in A \Rightarrow x \in B)$$

Definition 1.1.3 Let A and B be two sets, we say that A equal to B iff A is a subset of B and B is a subset of A. We symbolise

$$A = B \iff (A \subseteq B) \wedge (B \subseteq A).$$

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- If $B \subseteq A$ and $B \neq A$ then we say that B is a *proper subset* of A and we write $B \subset A$.
- A is not subset of B iff there exists an element x such that $x \in A$ and $x \notin B$, we symbolise

$$A \nsubseteq B \iff \exists x : \quad (x \in A) \wedge (x \notin B).$$

Theorem 1.1.1 For all sets A, B and C, we have

1. $A = A$;
2. If $A = B$, then $B = A$;
3. If $A = B$ and $B = C$, then $A = C$;

Theorem 1.1.2 For all sets A, B and C, we have

1. $A \subseteq A$ (reflexivity);
2. $A \subseteq B$ and $B \subseteq A$ imply $A = B$ (antisymmetry);
3. $A \subseteq B$ and $B \subseteq C$ imply$A \subseteq C$ (transitivity);

Definition 1.1.4 Let A and B be two non-empty sets

- The *complement of* A denoted by A^c is the set of all elements which do not belong to A, i.e.,

$$A^c \triangleq \{x : x \notin A\}.$$

- The *intersection* of A and B denoted by $A \cap B$ (A intersect B) is the set of all elements that belong to both A and B, i.e.,

$$A \cap B \triangleq \{x : (x \in A) \wedge (x \in B)\}.$$

- The *union* of A and B denoted by $A \cup B$ (A union B) is the set of all elements that belong either to A or to B or to both A and B, i.e.,

$$A \cup B \triangleq \{x : (x \in A) \vee (x \in B)\}.$$

- The *set difference* of A and B denoted by $A \setminus B$ (A minus B) is the set of all elements that belong to A and don't belong to B, i.e.,

$$A \setminus B \triangleq \{x : (x \in A) \wedge (x \notin B)\}.$$

- The *Cartesian product* of A and B denoted by $A \times B$ (A times (cross) B) is the set of all ordered pairs (x, y) with $x \in A$ and $y \in B$, i.e.,

$$A \times B \triangleq \{(x,y) : (x \in A) \wedge (y \in B)\}.$$

If $A = B$, then we denote the Cartesian product by $A \times A$ by A^2.

1.2 Mappings

Definition 1.2.1 Let X and Y be non-empty sets. A *mapping* (map or function) f from X to Y is an assignment (rule) that relates each element $x \in X$ with a **unique** point $y \in Y$ denoted $f(x)$ (xf, f_x or fx) (the *value* of f at the *argument* x or the *image* of x under f). We denote symbolically this assignment by $f : X \to Y$ or $X \xrightarrow{f} Y$ (f maps X to Y) where X is called the *domain* of f ($X \triangleq \mathrm{dom} f (\mathrm{dom}(f)) \triangleq \mathfrak{D}_f$) and Y is called the *codomain* of f. On occasion we use the symbol $x \mapsto f(x)$ which means that x maps to $f(x)$. And sometimes also

$$\begin{array}{rccc} f: & X & \to & Y \\ & x & \mapsto & f(x). \end{array}$$

The *range* (image) of f is the subset $f(X)$ ($\mathrm{Im}(f)$ or $\mathrm{Im} f$) of Y defined by

$$\mathrm{Im} f \triangleq \{f(x) : x \in X\}.$$

The *graph* of f denoted $\mathscr{G}_f$ is the set of ordered pairs $(x,y) \in X \times Y$ such that $y = f(x)$, i.e.,

$$\begin{aligned} \mathscr{G}_f &\triangleq \{(x,y) \in X \times Y : y = f(x)\}, \\ &= \{(x, f(x)) : x \in X\}. \end{aligned}$$

The set of all maps from X to Y is denoted by $\mathscr{F}(X,Y)$.

Definition 1.2.2 The *image* by a function $f : X \to Y$ of $A \subset X$, is the subset $f(A)$ of Y defined by

$$f(A) \triangleq \{f(x) : x \in A\},$$

The *inverse image* (preimage) by f of a subset B of Y is the subset $f^{-1}(B)$ of X

defined by

$$f^{-1}(B) \triangleq \{x \in X : f(x) \in B\}.$$

Definition 1.2.3 A map $f : X \to Y$ is said

- a *surjection* (surjective) iff $f(X) = Y$;
- an *injection* (injective) iff $\forall x_1, x_2 \in X, \quad x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$, in an equivalent manner, $\forall x_1, x_2 \in X, \quad f(x_1) = f(x_2) \implies x_1 = x_2$;
- a *bijection* (bijective) iff $\forall y \in Y, \exists! x \in X : \quad y = f(x)$, i.e., f is a surjection and an injection.

Definition 1.2.4 Let $f : X \to Y$ and $g : Y' \to Z$ ($Y \subseteq Y'$) be two functions. The *composition* of g and f is the function denoted by $g \circ f$ (g round f) and defined by

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y \subseteq Y' & \xrightarrow{g} & Z \\ \downarrow & & & & \downarrow \\ x & \longrightarrow & f(x) & \longrightarrow & g[f(x)]. \end{array}$$

Thus,

$$\begin{array}{cccc} g \circ f : & X & \to & Z \\ & x & \mapsto & g \circ f(x) = g[f(x)]. \end{array}$$

$g \circ f$ is the *composite* of g and f.

R In general, $f \circ g \neq g \circ f$

Theorem 1.2.1 Let $f : X \to Y$ and $g : Y \to Z$. Then

- $g \circ f$ is injective $\implies$ f is injective;
- $g \circ f$ is surjective $\implies$ g is surjective.

Theorem 1.2.2 Let $f : X \to Y$ and $g : Y \to Z$. Then,

1. if f and g are both injections, then $g \circ f$ is an injection.
2. if f and g are both surjections, then $g \circ f$ is a surjection.
3. if f and g are both bijections, then $g \circ f$ is a bijection.

Definition 1.2.5 Let $f : X \to Y$ be a bijection. Then there exists a unique function called the *inverse* function of f and is defined by $f^{-1} : Y \to X$ such that

$$\begin{cases} \forall x \in X, & f^{-1}[f(x)] = x \\ \forall y \in Y, & f\left[f^{-1}(y)\right] = y. \end{cases}$$

In other words, if we define the *identity function* of a set A as

$$\begin{array}{rccl} \mathrm{id}_A: & A & \to & A \\ & x & \mapsto & \mathrm{id}_A(x) = x. \end{array}$$

Then, the inverse function verifies $f^{-1} \circ f = \mathrm{id}_X$ and $f \circ f^{-1} = \mathrm{id}_Y$.

Theorem 1.2.3 If $f : X \to Y$ and $g : Y \to X$ are functions such that

$$f \circ g = \mathrm{id}_Y \text{ and } g \circ f = \mathrm{id}_X.$$

Then, f is a bijection and $f^{-1} = g$ (note that g is also a bijection and $g^{-1} = f$. Thus, f and g are each other's inverses.)

Theorem 1.2.4 If $f : X \to Y$ and $g : Y \to Z$ are bijections. Then,

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

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