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MATHEMATICS 1

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Contents

Part I

Real functions of one real variable

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1. Generalities

Contents

1.1 Some properties of $\mathbb{R}$

Definition 1.1.1 Let A be a non-empty subset of $\mathbb{R}$.

- We say that A is *bounded above* if $\exists M \in \mathbb{R} : \forall a \in A, \quad a \leq M$. M is called an *upper bound* (majorant) for A and it is not unique. We call the smallest upper bound *supremum* (least upper bound (lub)) of the set A and it is denoted by $\sup A$ ($\sup_{x \in A} x \triangleq \sup\{x : x \in A\}$).
- We say that A is *bounded below* if $\exists m \in \mathbb{R} : \forall a \in A, \quad m \leq a$. m is called a *lower*

bound (minorant) for A. We call the biggest lower bound *infemum* (greatest lower bound (glb)) of the set A and it is denoted by $\inf A$ ($\inf_{x\in A} x \triangleq \inf\{x : x \in A\}$).

- We say that A is *bounded* if a is bounded above and bounded below. If A is not a bounded set, then we say that A is *unbounded*.

R

(i) If $\sup A \in A$, then the $\sup A$ is called *maximum* of A and denoted by $\max A$ ($\max_{x\in A} x \triangleq \max\{x : x \in A\}$).
(ii) If $\inf A \in A$, then the $\inf A$ is called *minimum* of A and denoted by $\min A$ ($\min_{x\in A} x \triangleq \min\{x : x \in A\}$).
(iii) $\sup \emptyset \triangleq -\infty$ and $\inf \emptyset \triangleq \infty$

Theorem 1.1.1 Let A be an non-empty subset of $\mathbb{R}$.

i) If A is bounded above, then $\sup A$ exists and it is unique. If A contains a biggest element $\max A$, then $\sup A = \max A$.
ii) If A is bounded below, then $\inf A$ exists and it is unique. If A contains a smallest element $\min A$, then $\inf A = \min A$.

Theorem 1.1.2 Let A be an non-empty subset of $\mathbb{R}$.

i) If A is bounded above, then a real number M is called $\sup A$ iff

$$\begin{cases} M \text{ is an upper bound,} \\ \forall \varepsilon > 0, \exists x \in A : \quad x > M - \varepsilon. \end{cases}$$

ii) If A is bounded below, then a real number m is called $\inf A$ iff

$$\begin{cases} m \text{ is a lower bound,} \\ \forall \varepsilon > 0, \exists x \in A : \quad x < m + \varepsilon. \end{cases}$$

Definition 1.1.2 Let $x \in \mathbb{R}$. The *absolute value* of x is the nonnegative number $|x|$ defined by

$$|x| \triangleq \max\{x; -x\} \triangleq \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Properties 1.1.1

(a) $\forall a, b \in \mathbb{R}, \quad |a + b| \leq |a| + |b|;$ *(triangle inequality)*
(b) $\forall a, b \in \mathbb{R}, \quad ||a| - |b|| \leq |a - b|;$
(c) $\forall a, b \in \mathbb{R}, \quad |ab| = |a|\,|b|;$

Definition 1.1.3 Let $x \in \mathbb{R}$. The *floor* of x, denoted by $\lfloor x \rfloor$ ($[x], \text{floor}(x)$), is the greatest integer less than or equal to x, i.e.,

$$\lfloor x \rfloor \triangleq \max\{n \in \mathbb{Z} : n \leq x\}$$

Theorem 1.1.3 For all real number x, the floor of x is the unique integer that verifies

$$x - 1 < \lfloor x \rfloor \leq x.$$

Properties 1.1.2 *For all real number x, we have*

- $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$;
- $\forall n \in \mathbb{Z}, \quad \lfloor x + n \rfloor = \lfloor x \rfloor + n$;
- $\forall y \in \mathbb{R}, \quad \lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor + \varepsilon(x,y), \qquad$ *with* $\varepsilon(x,y) \in \{0;1\}$

1.2 Real functions on $\mathbb{R}$

1.2.1 Basic properties of real functions

Definition 1.2.1 A function $f : X \to Y$ is said

- *real function* if $Y \subseteq \mathbb{R}$;
- *function of one real variable* if $X \triangleq \mathscr{D}_f \subseteq \mathbb{R}$

Let f and g be real functions.

- $\forall \lambda \in \mathbb{R}, \quad \lambda f$ is the real function defined by $\forall x \in \mathscr{D}_f, \quad (\lambda f)(x) \triangleq \lambda f(x)$.
- $f + g$ is the real function defined by $\forall x \in \mathscr{D}_f \cap \mathscr{D}_g, \quad (f+g)(x) \triangleq f(x) + g(x)$.
- fg is the real function defined by $\forall x \in \mathscr{D}_f \cap \mathscr{D}_g, \quad (fg)(x) \triangleq f(x)g(x)$.
- f/g is the real function defined by $\forall x \in (\mathscr{D}_f \cap \mathscr{D}_g) \setminus \{x \in \mathbb{R} : g(x) = 0\}$, $(f/g)(x) \triangleq \frac{f(x)}{g(x)}$.
- $g \circ f$ is the real function defined by $\forall x \in \mathscr{D}_f \cap \left\{x \in \mathbb{R} : f(x) \in \mathscr{D}_g\right\}$, $(g \circ f)(x) \triangleq g[f(x)]$.
- The *restriction* of the function f on $A \subset \mathscr{D}_f$ is the function denoted by $f_{|_A}$ and is defined by $\forall x \in A, \quad f_{|_A}(x) \triangleq f(x)$.
- The *extension* of the function f on $A \supset \mathscr{D}_f$ is a function g such that $\forall x \in \mathscr{D}_f, \quad g_{|_{\mathscr{D}_f}}(x) = f(x)$.
- f is a *constant function* if $\exists c \in \mathbb{R} : \forall x \in \mathscr{D}_f, \quad f(x) = c$.

1.2.2 Parity and periodicity

Let $f \in \mathscr{F}\left(\mathscr{D}_f; \mathbb{R}\right)$.

- f is a *periodic* function if $\exists T \in \mathbb{R} : \forall x \in \mathscr{D}_f, \ \left[x + T \in \mathscr{D}_f\right] \wedge [f(x+T) = f(x)]$.

  R In general the least positive number T such that $\forall x \in \mathscr{D}_f, \quad f(x+T) = f(x)$ is called the *period* of the function f, and we can say f is T-periodic.

- f is an *even* function if $\forall x \in \mathscr{D}_f, \quad \left[-x \in \mathscr{D}_f\right] \wedge [f(-x) = f(x)]$.

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- f is an *odd* function if $\forall x \in \mathscr{D}_f, \quad [-x \in \mathscr{D}_f] \wedge [f(-x) = -f(x)]$.

R $\mathscr{D}_f$ is symmetric about the origin iff $\forall x \in \mathbb{R}, \quad x \in \mathscr{D}_f \implies -x \in \mathscr{D}_f$.

Theorem 1.2.1 Each function $f : \mathscr{D} \subseteq \mathbb{R} \to \mathbb{R}$ can be written as a sum of two real functions defined on $\mathscr{D}$, one is odd and the other is even.

1.2.3 Sens of variation

Definition 1.2.2 Let f and g be two functions with the same domain $\mathscr{D}$, then

$$f \begin{pmatrix} = \\ \leq \\ < \\ \geq \\ > \end{pmatrix} g \iff \forall x \in \mathscr{D}, \quad f(x) \begin{pmatrix} = \\ \leq \\ < \\ \geq \\ > \end{pmatrix} g(x).$$

Definition 1.2.3 Let f be real function on $\mathscr{D}_f \subseteq \mathbb{R}$.

- f is called *nondecreasing (increasing)* or *(increasing (strictly increasing))* on $\mathscr{I} \subset \mathscr{D}$ if $\forall x_1, x_2 \in \mathscr{I}, \quad x_1 < x_2 \implies f(x_1) \leq (<) f(x_2)$. If $\mathscr{I} = \mathscr{D}_f$, then we say only f is *(strictly) increasing*
- f is called *nonincreasing (decreasing)* or *(decreasing (strictly decreasing))* on $\mathscr{I} \subset \mathscr{D}$ if $\forall x_1, x_2 \in \mathscr{I}, \quad x_1 < x_2 \implies f(x_1) \geq (>) f(x_2)$. If $\mathscr{I} = \mathscr{D}_f$, then we say only f is *(strictly) decreasing*
- f is called *(strictly) monotone* ((strictly) monotonic) on $\mathscr{I} \subset \mathscr{D}_f$ if f is (strictly) increasing or (strictly) decreasing on $\mathscr{I} \subset \mathscr{D}_f$. If $\mathscr{I} = \mathscr{D}_f$, then we say only f is *(strictly) monotone*

Theorem 1.2.2 A real function of one real variable f is bijective on a subset $\mathscr{I} \subset \mathscr{D}_f$ iff f is (strictly) monotone on $\mathscr{I}$.

R If $\mathscr{I} = \mathscr{D}_f$ we say only f is bijective.

1.2.4 Bounded functions

Definition 1.2.4 Let $f : \mathscr{D}_f \to \mathbb{R}$ be a function. We say that

- f is *bounded above* on $\mathscr{D}_f$ if there is $M \in \mathbb{R}$ such that $\forall x \in \mathscr{D}_f, \quad f(x) \leq M$. M is called an *upper bound* for f. The smallest upper bound is called the *least upper bound (lub)* or (*supremum*) and is denoted by $\sup f$ ($\sup_{x \in \mathscr{D}_f} f(x)$ or $\sup \{f(x) : x \in \mathscr{D}_f\}$). If there exists $a \in \mathscr{D}_f$ such that $f(a) = \sup f$, then we say that f *attains its upper bound* and $f(a)$ is called also *maximum*, we can write $f(a) = \max f$ ($\max_{x \in \mathscr{D}_f} f(x)$ or $\max \{f(x) : x \in \mathscr{D}_f\}$).
- f is *bounded below* on $\mathscr{D}_f$ if there is $m \in \mathbb{R}$ such that $\forall x \in \mathscr{D}_f, \quad m \leq f(x)$.

m is called a *lower bound* for f. The biggest lower bound is called the *greatest lower bound (glb)* or (*infemum*) and is denoted by $\inf f$ ($\inf_{x\in\mathscr{D}_f} f(x)$ or $\inf\{f(x) : x \in \mathscr{D}_f\}$). If there exists $b \in \mathscr{D}_f$ such that $f(b) = \inf f$, then we say that f *attains its lower bound* and $f(b)$ is called also *minimum*, we can write $f(b) = \min f$ ($\min_{x\in\mathscr{D}_f} f(x)$ or $\min\{f(x) : x \in \mathscr{D}_f\}$).

- f is *bounded* on $\mathscr{D}_f$ if it is bounded above on $\mathscr{D}_f$ and also bounded below on $\mathscr{D}_f$.

R f is bounded iff $\exists\,\delta > 0 : \forall x \in \mathscr{D}_f, \quad |f(x)| \leq \delta$. δ is called the *bound* for the absolute value of f.

f is a bounded function iff $f(\mathscr{D}_f)$ is a bounded set.

1.2.5 The floor function

The *floor* function f is defined on $\mathbb{R}$ by $f : x \mapsto f(x) = \lfloor x \rfloor$.

1.2.6 Elementary functions

Absolute value function

The *absolute value* function f is defined on $\mathbb{R}$ by $f : x \mapsto f(x) = |x|$.

Power functions

Let r be a non-zero rational number. The *power function* f is the function $f : x \mapsto f(x) = x^r$.

- If $r \in \mathbb{N}$, then the definition domain of f is $\mathbb{R}$. If n is even (resp. odd), the f is even (resp. odd) and f is increasing on $[0;\infty)$, i.e., f is a bijection from $[0;\infty)$ on it self.
- If $r \in \mathbb{Z}_-$, then the definition domain of f is $\mathbb{R}\setminus\{0\}$. If n is even (resp. odd), the f is even (resp. odd) and f is decreasing on $(0;\infty)$, i.e., f is a bijection from $(0;\infty)$ on it self.
- If $r = \frac{p}{q}$ with $CGD\{p;q\} = 1$, then
  a) q is even, then the definition domain of f is $[0;\infty)$.
  b) q is odd, then the definition domain of f is $\mathbb{R}$.
  c) p is even (resp. odd), then f is even (resp. odd). The restriction of f on $[0;\infty)$ on it self is a bijection.
- If $r = -\frac{p}{q}$ with $CGD\{p;q\} = 1$, then
  a) q is even, then the definition domain of f is $(0;\infty)$.
  b) q is odd, then the definition domain of f is $\mathbb{R}\setminus\{0\}$.
  c) p is even (resp. odd), then f is even (resp. odd). The restriction of f on $(0;\infty)$ on it self is a bijection.

Properties 1.2.1 *Let $x, y \in \mathbb{R}\setminus\{0\}$ and $p, q \in \mathbb{Q}_+\setminus\{0\}$*

- $x^p x^q = x^{p+q}$;
- $\frac{x^p}{x^q} = x^{p-q}$;
- $(x^p)^q = x^{pq}$;
- $x^p y^p = (xy)^p$;
- $x^0 = 1$;

Polynomial functions

Let n be a natural number and $a_k,\ (k=\overline{0,n})$ be real numbers. The *polynomial function* is the function $x \mapsto \sum_{k=0}^{n} a_k x^k$. If $a_n \neq 0$, then n is called the *degree* (d°) of the polynomial function. The definition domain of the polynomial functions is $\mathbb{R}$. The constants $a_k,\ (k=\overline{0,n})$ are called the coefficients of the function polynomial.

Properties 1.2.2 *Let P and Q be two polynomial functions, n and m be two natural numbers. Put $\mathrm{d}^\circ P = n$ and $\mathrm{d}^\circ Q = m$.*

- *If $n=1$, then P is called* linear function.
- *If $n=2$, then P is called* quadratic function.
- *If $n=3$, then P is called* cubic function.
- *$P \pm Q$ is a polynomial function. If $n \neq m$, then $\mathrm{d}^\circ(P \pm Q) = \max\{n;m\}$*
- *$P \cdot Q$ is a polynomial function of degree $n+m$.*

Theorem 1.2.3 — Real Fundamental Theorem of Algebra. Every nonzero polynomial with real coefficients can be factored as a finite product of linear polynomials and quadratic polynomials with negative discriminants.

Corollary 1.2.4 Let $l,k \in \mathbb{N}$. For all $i \in [\![0;l]\!],\ j \in [\![0;k]\!]$, let $n_i,\ m_j$ be natural numbers, $a_i,\ p_j,\ q_j$ be real numbers such that $p_j^2 - 4q_j < 0$ and let R be a non-constant polynomial function. Then
$\forall x \in \mathbb{R}$,

$$R(x) = \left[\begin{array}{l} \prod_{i=0}^{l} (x-a_i)^{n_i} \\ \vee \\ \prod_{j=0}^{k} \left(x^2+p_j x+q_j\right)^{m_j} \\ \vee \\ \prod_{i=0}^{n} (x-a_i)^{n_i} \prod_{j=0}^{k} \left(x^2+p_j x+q_j\right)^{m_j} \end{array}\right.$$

Rational functions

Let n and m be two natural numbers, P and Q be two polynomial functions of degrees n and m respectively. The *rational function* is the function $x \mapsto \frac{P(x)}{Q(x)}$ (For an x well defined, $P(x)/Q(x)$ is called a rational fraction). The *degree* of the rational function is denoted by $n-m$. The definition domain of P/Q is $\mathbb{R} \setminus \{x \in \mathbb{R} : Q(x) = 0\}$. If n<m, then the ration function P/Q is called *proper*, and if $n \geq m$, then P/Q is called *improper rational function*.

Theorem 1.2.5 Any improper rational function P/Q can be decomposed as the sum of a polynomial function and a proper rational function, i.e.,

$$\frac{P}{Q} = R + \frac{S}{T},$$

with R is a polynomial function and S/T is a proper rational function ($\mathrm{d}^\circ S < \mathrm{d}^\circ T$).

Let $a,\ A, \alpha,\ \beta,\ p,\ q$ be real numbers such that $p^2-4q<0$. We call *partial function* any proper rational function that is written as

- $x \mapsto \frac{A}{x-a}$; (linear factor)
- $x \mapsto \frac{\alpha x+\beta}{x^2+px+q}$; (quadratic irreducible factor)

For $x\in\mathbb{R}$ well defined the numbers $\frac{A}{x-a}$ and $\frac{\alpha x+\beta}{x^2+px+q}$ are called *partial fractions*.

Theorem 1.2.6 — Partial Fraction Decomposition. Every proper rational function can be decomposed as the sum of finitely many rational functions of the form

$$\frac{A}{(x-a)^n} \text{ and } \frac{\alpha x+\beta}{(x^2+px+q)^m} \quad (\text{ for } x\in\mathbb{R} \text{ well defined}),$$

where A, a and α,β,p,q are real numbers such that $p^2-4q<0$; and n,m are natural numbers, i.e., let P/Q be a proper fractional function. Then, under the hypothesis of Corollary 1.2.4 we get
$\forall x\in\mathscr{D}_{P/Q}$,

1. If $Q(x)=\prod_{i=1}^{l}(x-a_i)^{n_i}$, then
$\exists A_{i,j}\in\mathbb{R}\ (i=\overline{1,l};\ j=\overline{1,\max_{i\in[\![1;l]\!]} n_i})$:
$$\frac{P(x)}{Q(x)}=\sum_{i=1}^{l}\sum_{j=1}^{n_i}\frac{A_{i,j}}{(x-a_i)^j}.$$
2. If $Q(x)=\prod_{j=1}^{k}\left(x^2+p_jx+q_j\right)^{m_j}$, $(p_j^2-4q_j<0)$, then
$\exists\, \alpha_{i,j},\ \beta_{i,j}\in\mathbb{R}\ (i=\overline{1,k};\ j=\overline{1,\max_{i\in[\![1;k]\!]} m_i})$:
$$\frac{P(x)}{Q(x)}=\sum_{i=1}^{k}\sum_{j=1}^{m_i}\frac{\alpha_{i,j}x+\beta_{i,j}}{(x^2+p_ix+q_i)^j}.$$
3. If $Q(x)=\prod_{i=1}^{l}(x-a_i)^{n_i}\prod_{j=1}^{k}\left(x^2+p_jx+q_j\right)^{m_j}$, then
$$\exists A_{i,j}\in\mathbb{R}\ (i=\overline{1,l};\ j=\overline{1,\max_{i\in[\![1;l]\!]} n_i}) \text{ and}$$
$$\exists\, \alpha_{i,j},\ \beta_{i,j}\in\mathbb{R}\ (i=\overline{1,k};\ j=\overline{1,\max_{i\in[\![1;k]\!]} m_i}) :$$
$$\frac{P(x)}{Q(x)}=\sum_{i=1}^{l}\sum_{j=1}^{n_i}\frac{A_{i,j}}{(x-a_i)^j}+\sum_{i=1}^{k}\sum_{j=1}^{m_i}\frac{\alpha_{i,j}x+\beta_{i,j}}{(x^2+p_ix+q_i)^j}.$$

Exponential and logarithmic functions

Let a be a positive number. The *exponential function* is the function $f: x\mapsto f(x)=a^x$. The definition domain of f is $\mathbb{R}$.

- If $a>1$, then f is increasing.

- If $a<1$, then f is decreasing.
- f is a bijection from $\mathbb{R}$ to $(0;\infty)$.
- The natural exponential function is defined by $x\mapsto e^x$, where $e\approx 2.71828$ is the Euler number.

Properties 1.2.3 *Let a,b be positive real numbers and x,y be any real numbers. Then*

- $a^0=1$
- $a^{-x}=\frac{1}{a^x}$
- $a^{x+y}=a^xa^y$
- $a^{x-y}=\frac{a^x}{a^y}$
- $(a^x)^y=a^{xy}$
- $(ab)^x=a^xb^x$

The inverse of the exponential function $x\mapsto a^x$ is called the *logarithm to the base a function* and defined for all $x\in(0;\infty)$ as the solution of the equation $a^x=b$ (a,b are positive parameters) ans is denoted by $g:x\mapsto g(x)=\log_a x$. The domain of definition of g is $(0;\infty)$.

- If $a>1$, then g is increasing.
- If $a<1$, then f is decreasing.
- f is a bijection from $\mathbb{R}$ to $(0;\infty)$.
- The natural logarithmic function is defined by $x\mapsto \ln x\triangleq\log_e x$.

Properties 1.2.4 *Let a,b be positive real numbers and x,y be any positive real numbers. Then*

- $\log_a(a^x)=a^{(\log_a x)}=x$.
- $\log_a 1=0$.
- $\log_a(xy)=\log_a x+\log_a y$.
- $\log_a(x/y)=\log_a x-\log_a y$.

Trigonometric functions and their inverses

The *trigonometric functions* are the functions defined by $x\mapsto\cos x$, $x\mapsto\sin x$, $x\mapsto\tan x$... .

Properties 1.2.5

- *The domain of definition of* cos *is $\mathbb{R}$, it is an even function and a $2\pi-$periodic function. The* cos *is a bijection from $[0;\pi]$ to $[-1;1]$.*
- *The domain of definition of* sin *is $\mathbb{R}$, it is an odd function and a $2\pi-$periodic function. The* sin *is a bijection from $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$ to $[-1;1]$.*
- *The domain of definition of* tan *is $\mathbb{R}\setminus\left\{\frac{\pi}{2}+k\pi:k\in\mathbb{Z}\right\}$, it is an odd function and a $\pi-$periodic function. The* tan *is a bijection from $\left]-\frac{\pi}{2};\frac{\pi}{2}\right[$ to $\mathbb{R}$.*

We define the inverse of the trigonometric functions as:

- The *arccosine* function is the function $\arccos:[-1;1]\to[0;\pi]$, $x\mapsto\arccos x$ such that $\forall\theta\in[0;\pi],\quad \cos(\arccos\theta)=\arccos(\cos\theta)=\theta$.
- The *arcsine* function is the function $\arcsin:[-1;1]\to\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$, $x\mapsto\arcsin x$ such that $\forall\theta\in\left[-\frac{\pi}{2};\frac{\pi}{2}\right],\quad \sin(\arcsin\theta)=\arcsin(\sin\theta)=\theta$.
- The *arctangent* function is the function $\arctan:\mathbb{R}\to\left]-\frac{\pi}{2};\frac{\pi}{2}\right[$, $x\mapsto\arctan x$ such that $\forall\theta\in\left]-\frac{\pi}{2};\frac{\pi}{2}\right[,\quad \tan(\arctan\theta)=\arctan(\tan\theta)=\theta$.

Hyperbolic functions and their inverses

The *hyperbolic functions* are the functions defined by : $\cosh:x\mapsto\cosh x\triangleq\frac{e^x+e^{-x}}{2}$ and $\sinh:x\mapsto\sinh x\triangleq\frac{e^x-e^{-x}}{2}$... .

Conclusion

Elementary functions of a single real variable x include:

- All functions obtained by adding, subtracting, multiplying or dividing a finite number of any of the previous functions.
- All functions obtained by root extraction of a polynomial with coefficients in elementary functions.
- All functions obtained by composing a finite number of any of the previously listed functions.