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Module: Algebra 2
L1 Math+MI

Tutorial sheet N°2 : Linear maps

Exercise 1

We consider the map f defined by :

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x,y) \longmapsto \left(x-y,\ -3x+3y\right)$$

(1) Show that f is a linear map.
(2) Determine the image of the vector $v=(1,1)$ by f.
(3) Is there a vector $u=(x,y)\in\mathbb{R}^2$ such that : $f(u)=\left(1,\ 0\right)$?
(4) Give a basis of $\ker(f)$ and a basis of $\mathrm{Im}(f)$.
(5) Show by two methods that f is neither injective nor surjective.

Exercise 2

We consider the map T defined by :

$$T : \mathbb{R}_2[X] \longrightarrow \mathbb{R}_3[X]$$
$$P \longmapsto (X^2+1)P'$$

(1) Show that T is a linear map.
(2) Determine $\ker(T)$, the kernel of T and deduce $r(T)$, the rank of T.
(3) Is the map T injective? Surjective?.

Exercise 3 **(Homework)**

We consider the map f defined by :

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$
$$(x,y,z) \longmapsto (x-y, y+z)$$

(1) Show that f is a linear map
(2) Show that $\ker(f)$ is a subspase of $\mathbb{R}^3$.
(3) Determine $\dim(\ker(f))$. Is f injective?
(4) Deduce $r(f)$, the rank of f. Is the map f surjective?

Exercise 4 **(Supplementary)**

We consider the map f defined by : $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$(x,y,z) \longmapsto \left(-2x+y+z,\ x-2y+z, x+y-2z\right)$$

(1) Show that f is a linear map.
(2) Give a basis of $\ker(f)$ and deduce the rank of f.
(3) Give a basis of $\mathrm{Im}(f)$.

Exercise 5 **(Supplementary)**

Let f be a map defined by $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$(x,y) \longmapsto \left(x+y,\ x, y\right)$$

(1) Show that f is a linear map.
(2) Determine $\ker(f)$.
(3) Determine $r(f)$.
(4) Is the map f injective? Surjective?.

INDICATIONS FOR SUPPLEMENTARY EXERCISES

Exercise 4

(1)

$$u_1 = (x_1,y_1,z_1), u_2 = (x_2,y_2,z_2)$$

$$u_1 + u_2 = (x_1 + x_2, y_1 + y_2, z_1 + z_2) = (X,Y,Z)$$

$$f(u_1 + u_2) = f(X,Y,Z) = (-2X + Y + Z, X - 2y + Z, X + Y - 2Z)$$

$= ([-2x_1 + y_1 + z_1] + [-2x_2 + y_2 + z_2], [x_1 - 2y_1 + z_1] + [x_2 - 2y_2 + z_2], [x_1 + y_1 - 2z_1] + [x_2 + y_2 - 2z_2])$

$= f(u_1 + u_2) = f(u_1) + f(u_2)$

$$f(\alpha u) = \left(-2\alpha x + \alpha y + \alpha z,\ \alpha x - 2\alpha y + \alpha z, \alpha x + \alpha y - 2\alpha z\right)$$

$$= \alpha\left(-2x + y + z,\ x - 2y + z, x + y - 2z\right)$$

(2) $u = (x,y,z) \in \ker(f) \Rightarrow f(x,y,z) = (0,0,0) \Rightarrow \begin{cases} -2x + y + z \\ x - 2y + z \\ x + y - 2z \end{cases} \Rightarrow x = y, z = y$

$\ker(f) = Vect\{(1,1,1)\}$

$r(f) = \dim \mathbb{R}^3 - \dim(\ker(f)) = 2$

(3) $\mathrm{Im}(f) = Span\{(-2,1,1),(1,-2,1),(1,1,-2)\} = Span\{(1,-2,1),(1,1,-2)\}$

because $(-2,1,1) = -(1,-2,1) - (1,1,-2)$

Exercie 5

(2) $u = (x,y) \in \ker(f) \Rightarrow \begin{cases} x + y = 0 \\ x = 0 \\ y = 0 \end{cases} \Rightarrow x = y = 0$

$\ker(f) = \{(0,0)\}$

(3) $r(f) = \dim \mathbb{R}^2 - \dim(\ker(f)) = 2$

or $f(x,y) = \left(x + y,\ x, y\right) = x(1,1,0) + y(1,0,1)$

$\mathrm{Im}(f) = Vect\{(1,1,0),(1,0,1)\}$

$r(f) = 2$

(3) f is inj

$\mathrm{Im}(f) \neq \mathbb{R}^3$ not surj