Abou Bekr Belkaid University - Tlemcen
Faculty of Sciences
Department of Mathematics

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Module: Algebra 2
L1 Math+MI

Tutorial sheet N°1 : Vector spaces

Exercise 1

(1) Let $E = \mathbb{R}^{+*}$ be provided with a binary operation $\oplus$ defined by :
$\forall a,b \in E : a \oplus b = ab$, and an external law $\otimes$ such that:
$\forall a \in E, \forall \lambda \in \mathbb{R} : \lambda \otimes a = a^{\lambda}$.
- Show that $(E,\oplus,\otimes)$ is a $\mathbb{R}$-vector space.
(2) Let U be the set of sequences $(u_n)_{n\in\mathbb{N}}$ such that $\lim\limits_{n\to+\infty} u_n = 0$.
- Show that $(U,+,.)$ is a $\mathbb{R}$-vector space.

Exercise 2

Among the sets F say which are subspaces of E in each case :
(1) $E = \mathbb{R}^3$, $F = \left\{(x,y,z) \in \mathbb{R}^3;\ x + 2y + 3z = 0\right\}$. (2) $E = \mathbb{R}^2$, $F = \left\{(x,y) \in \mathbb{R}^2;\ 2x + 3y = 0\right\}$
(3) $E = \mathbb{R}^2$, $F = \left\{(x,y) \in \mathbb{R}^2;\ x^2 + y^2 = 4\right\}$, (4) $E = \mathbb{R}[X]$, $F = \left\{P \in \mathbb{R}[X];\ \deg(P) = 4\right\}$
(5) $E = \mathbb{R}[X]$, $F = \mathbb{R}_4[X] = \left\{P \in \mathbb{R}[X];\ \deg(P) \leq 4\right\}$
(6) $E = \mathbb{R}^{\mathbb{N}}$ set of real sequences, $F = \left\{(u_n) \in E :\ f \text{ is increasing } \right\}$
(7) $E = \mathcal{F}(\mathbb{R},\mathbb{R})$, $F = \left\{f \in E; f(1) = 0\right\}$, (8) $E = \mathcal{F}(\mathbb{R},\mathbb{R})$, $F = \left\{f \in E; f(0) = 1\right\}$.

Exercise 3

Let $u = (1,2,3)$ and $v = (2,3,1)$ be two vectors of $\mathbb{R}^3$.
(1) Write the vectors $w_1 = (1,3,8)$ and $w_2 = (2,4,5)$ as linear combinations of u and v.
(2) Determine the real number k so that the vector $w = (1,k,-2)$ is a linear combination of u and v.
(3) Determine conditions on the real numbers a, b and c so that the vector $W = (a,b,c)$ can be written as a linear combination of the vectors u and v.

Exercise 4

Let $U = \left\{(x,y,z) \in \mathbb{R}^3;\ x = y = z\right\}$ and $W = \left\{(x,y,z) \in \mathbb{R}^3;\ x = 0\right\}$
(1) Show that U and W are subspaces of $\mathbb{R}^3$.
(2) Show that : $\mathbb{R}^3 = U \oplus W$

Exercise 5

(1) Show that : $F = \left\{(X-1)^3, (X-1)^2, (X-1), 1\right\}$ is a basis of $\mathbb{R}_3[X]$.
(2) Find the components of the polynomial $P(X) = 3X^3 - 4X^2 + 2X - 5$ in this basis.

Exercise 6

Let $F = Vect\{u_1,u_2\}$ be the subspace of $\mathbb{R}^3$ spanned by $\{u_1,u_2\}$ where
$u_1 = (1,0,-1)$, $u_2 = (0,1,-1)$ et $G = \{(x,y,z) \in \mathbb{R}^3 : x - y = x + z = 0\}$.
(1) Show that the vectors u_1 and u_2 are linearly dependent.
(2) Show that G is a subspace of $\mathbb{R}^3$.
(3) Show that if $(x,y,z) \in F$ then $x + y + z = 0$.
(4) Determine a basis for F and a basis for G and deduce $\dim(F)$ and $\dim(G)$.
(5) Determine $F \cap G$ and deduce that $\mathbb{R}^3 = F \oplus G$.

SUPPLEMENTARY EXERCISES

Exercise 7

(1) Show that $S = \{(1,1,1),(1,1,0),(1,0,0)\}$ is abasis of $\mathbb{R}^3$.

(2) Find the components of $v = (1,1,2)$ and $w = (2,3,4)$ in the standard basis of $\mathbb{R}^3$and in the S basis.

(3) Determine the components of the standardl basis vectors in the S basis.

Exercise 8

Let

$U_1 = \{(a,b,c) \in \mathbb{R}^3 : a = c\}, U_2 = \{(a,b,c) \in \mathbb{R}^3 : a+b+c = 0\}$ and $U_3 = \{(0,0,c) : c \in \mathbb{R}\}$

(1) Show that : $U_i,\ i = 1,3$ are subspaces of $\mathbb{R}^3$.

(2) Show that : $\mathbb{R}^3 = U_1 + U_2, \mathbb{R}^3 = U_2 + U_3, \mathbb{R}^3 = U_1 + U_3$.

(3) Determine $\dim U_1,\ \dim U_2$ and $\dim U_3$.

(4) In which case the sum is direct.

Exercise 9

Let $A = \{(x+y, y-3x, x) : x,y \in \mathbb{R}\}$ and $B = \{(x,y,z) \in \mathbb{R}^3 : y = -2x = -2z\}$.

(1) Check that A and B are two subspaces of $\mathbb{R}^3$.

(2) Determine $\dim A$ and $\dim B$.

(3) Show that $A \oplus B = \mathbb{R}^3$.

Exercise 10

Let $\mathbb{R}_1[X]$ be the vector space of polynomials of degree less than or equal to 1 and $B = \{1,x\}$ its standard basis.

(1) Check that $\forall x \in \mathbb{R} : a + bx = \frac{a-b}{2}(1-x) + \frac{a+b}{2}(1+x)$

(2) Determine the real numbers α and β such that : $\forall x \in \mathbb{R},\ \alpha(1-x) + \beta(1+x) = 0$.

(3) Deduce that : $C = \{1-x, 1+x\}$ is a basis of $\mathbb{R}_1[X]$.

(4) Determine the components of the vector $P(x) = 2x+3$, as well as the vectors of the standard basis B in the basis C.

Exercise 11

Let $E = \mathbb{R}_1[X],\ E_1 = \{P \in E;\ P(-X) = P(X)\}$ and $E_2 = \{P \in E;\ P(-X) = -P(X)\}$.

(1) Show that E_1 and E_2 are subspaces of E.

(2) Use two methods to show that $E = E_1 \oplus E_2$.

Exercise 12

Let $\mathcal{F}(\mathbb{R},\mathbb{R})$ be the subspace over $\mathbb{R}$ of functions from $\mathbb{R}$ into $\mathbb{R}$.

Let $\wp(\mathbb{R},\mathbb{R})$ and $\Im(\mathbb{R},\mathbb{R})$ be the subspaces of $\mathcal{F}(\mathbb{R},\mathbb{R})$ of even and odd functions from $\mathbb{R}$ into $\mathbb{R}$ defined by :

$\wp(\mathbb{R},\mathbb{R}) = \{f \in \mathcal{F}(\mathbb{R},\mathbb{R}) : \forall x \in \mathbb{R} : f(-x) = f(x)\}$,

$\Im(\mathbb{R},\mathbb{R}) = \{f \in \mathcal{F}(\mathbb{R},\mathbb{R}) : \forall x \in \mathbb{R} : f(-x) = -f(x)\}$

Let $f \in \mathcal{F}(\mathbb{R},\mathbb{R})$. We define two functions p and i of $\mathcal{F}(\mathbb{R},\mathbb{R})$ by :

$\forall x \in \mathbb{R} : p(x) = \frac{1}{2}[f(x) + f(-x)],\ i(x) = \frac{1}{2}[f(x) - f(-x)]$.

(1) Show that p is even and i is odd. Deduce that $\mathcal{F}(\mathbb{R},\mathbb{R}) = \wp(\mathbb{R},\mathbb{R}) + \Im(\mathbb{R},\mathbb{R})$.

(2) Show that $\mathcal{F}(\mathbb{R},\mathbb{R}) = \wp(\mathbb{R},\mathbb{R}) \oplus \Im(\mathbb{R},\mathbb{R})$.