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Module: Algebra 2
L1 Math+MI

T.D N°02 : Polynomials and rational fractions

Exercise 1

(1) Calculate, develop and order according to decreasing powers, the following polynomial :

$$P(x) = (x^3 - x^2 + 2x - 1)(2x^2 - 3x + 3) + (1 - 2x + 2x^3)(2 + x - x^2).$$

(2) Perform the euclidean division in $\mathbb{R}[X]$ of $P(x)$ by $2x^2 - 3x + 3$.

(3) Carry out the division according to increasing powers to order 3 in $\mathbb{R}[X]$ of $P(x)$ by $1 - 2x + 2x^3$.

Exercise 2

(I) Let $P = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} \ldots + a_2 x^2 + a_1 x + a_0$, be a polynomial of degree $n \geq 1$ with coefficients in $\mathbb{Z}$.

(1) Prove that if P admits a root in $\mathbb{Z}$, then this divides a_0.

(2) Do the polynomials $x^3 - \quad x^2 - 109x - 11$ and $x^{10} + x^5 + 1$ have roots in $\mathbb{Z}$?

(II) Let $P(x) = x^5 - 3x^4 + 4x^3 - 4x^2 + 3x - 1$

(1) Show that P admits an integer root and determine its order of multiplicity.

(2) Factor the polynomial P into $\mathbb{R}[X]$, as a product of irreducible polynomials.

Exercise 3

(1) Calculate, develop and order according to increasing powers, the following polynomial:

$$P(x) = (1 + x)(x^3 + 1) + (2 + x)(x - 1).$$

(2) Factor in $\mathbb{R}[X]$ into the product of irreducible polynomials, the following polynomial:

$$Q(x) = x^3 + x^2 + x + 1 .$$

(3) Do the Euclidean division of $P(x)$ by $Q(x)$.

(4) Carry out the division according to the increasing powers to order 2 of $P(x)$ by $Q(x)$.

(5) Decompose the following rational fraction into simple elements:

$$F(x) = \frac{P(x)}{Q(x)}$$

Exercise 4

Decompose the following rational fractions into simple elements:

$$F_1(x) = \frac{1}{x^3 + 3x^2 + 2x}, \ F_2(x) = \frac{3x^4}{x^3 - 1},$$

$$F_3(x) = \frac{x^3 + 1}{(x - 1)^3}$$

SUPPLEMENTARY EXERCISES

Exercise 1

(1)Determine the polynomial P of degree 2 verifying:

$$P(0) = 0,\ \forall x \in \mathbb{R} : P(x) - P(x-1) = x........(*)$$

(2) Using the relation $(*)$, calculate the sum

$$S = 1 + 2 + 3 + \ldots + n$$

Exercise 2

Let $A(x) = x^4 - 3x^3 - x^2 + 8x - 4$.

(1) Show that $x_0 = 2$ is a double root of the polynomial A, and deduce the other roots.

(2) Factor $A(x)$ into $\mathbb{R}[X]$.

Exercise 3

Factor in $\mathbb{R}[X]$, the following polynomials:

$$P_1(x) = x^4 - 1,\ P_2(x) = x^6 + 1,\ Q_1(x) = x^6 + 2x^4 - 2x^2 + 1$$

$$Q_2(x) = x^{10} + x^5 + 1,\ R_1(x) = x^{12} - 1,\ R_2(x) = x^9 + x^6 + x^3 + 1$$

Exercise 4

Let $A(x) = x^3 - 2x^2 + x - 2$

(1) Show that A admits a simple integer root α.

(2) Factor $A(x)$ into $\mathbb{R}[X]$.

Exercise 5

What is the remainder of the Euclidean division of the polynomial

$$P(x) = (x+1)^n - x^n - 1 \text{ by}$$

$$(i)\ Q_1(x) = x^2 - 3x + 2$$

$$(ii)\ Q_2(x) = x^2 - 2x + 1$$

Exercise 6

Consider the following polynomial:

$$Q(x) = x^3 - x^2 - x + 1.$$

(1) Verify that $x_0 = 1$ is a double root of the polynomial Q.

(2) Factor in $\mathbb{R}[X]$ into the product of irreducible polynomials, the polynomial Q.

(3) Determine the polynomial P whose quotient and the remainder of its Euclidean division by Q are :

$$A(x) = x \text{ and } R(x) = 4 \text{ respectively.}$$

(4) Carry out the division according to the increasing powers to order 2 of P by Q.

(5) Decompose the following rational fraction into simple elements in $\mathbb{R}[X]$:

$$F(x) = \frac{P(x)}{Q(x)}$$

Exercise 7

Decompose the following rational fractions into simple elements:

$$F_1(x) = \frac{1}{x^2 - 3x + 2},\ F_2(x) = \frac{1}{x^3 - 3x^2 + 2}$$

$$F_3(x) = \frac{x^5 - 4x^4 + 5x^3 + x^2 - 4x + 5}{x^3 - 6x^2 - 11x - 6},\ F_4(x) = \frac{x^3}{x^4 - 1}$$