Abou Bekr Belkaid University - Tlemcen
Faculty of Sciences
Department of Mathematics

Academic year: 2023 - 2024
Module : Algebra 2
L1 Math+M1

Remedial exam

Exercise 1 (7 pts)

Let $E = Span\{(1,1,1)\}$ and $F = \{(x,y,z) \in \mathbb{R}^3 : x + y - z = 0\}$
(1) Show that F is a subspace of $\mathbb{R}^3$ and find its dimension.
(2) Find $E \cap F$
(3) Show that $\mathbb{R}^3 = E \oplus F$.

Exercise 2 (5 pts)

We consider the map F defined by :

$$F : \mathbb{R}_2[X] \to \mathbb{R}_2[X]$$
$$P \mapsto (X+1)P'$$

(1) Show that F is a linear map.
(2) Determine $\ker(F)$, the kernel of F and deduce $r(F)$, the rank of F.
(3) Is the map F injective? Surjective?

Exercise 3 (8 pts)

Let $M = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 5 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 5 \end{pmatrix}$

(1) Determine the matrices $A = {}^tM.M$ and $B = A + I_3$, where ${}^tM$ is the transpose matrix of M and I_3 is the unit (identity) matrix of order 3.
(2) Calculate $\det(B)$.
(3) Calculate $B.C$ and $C.B$.
(4) Solve the following system (S) by two methods : (matrix inversion method and Cramer method)

$$(S) \begin{cases} 2x + y = 3 \\ x + 3y + z = 5 \\ y + 2z = 3 \end{cases}$$

(5) Determine the values of $\lambda \in \mathbb{R}$ so that : $\det(B - \lambda I_3) = 0$.

Good luck

Université Abou Bekr Belkaid - Tlemcen
Faculté des sciences
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Année universitaire: 2023 - 2024
Module: Algèbre 2
L1 Math+MI

Examen de rattrapage

Exercice 1 (7 pts)

Soient $E = Vect\{(1,1,1)\}$ et $F = \{(x,y,z) \in \mathbb{R}^3 : x+y-z=0\}$

(1) Montrer que F est un sous espace vectoriel de $\mathbb{R}^3$ et déterminer sa dimension.
(2) Trouver $E \cap F$
(3) Montrer que $\mathbb{R}^3 = E \oplus F$.

Exercice 2 (5 pts)

On considère l'application F définie par :

$$F : \mathbb{R}_2[X] \to \mathbb{R}_2[X]$$
$$P \mapsto (X+1)P'$$

(1) Montrer que F est linéaire.
(2) Déterminer $\ker(F)$, le noyau de F et en déduire $rg(F)$, le rang de F.
(3) F est-elle injective? Surjective?.

Exercice 3 (8 pts)

Soient $M = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ et $C = \begin{pmatrix} 5 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 5 \end{pmatrix}$

(1) Déterminer les matrices $A = {}^tM.M$ et $B = A + I_3$, où ${}^tM$ est la matrice transposée de M et I_3 est la matrice unité d'ordre 3.
(2) Calculer $\det(B)$.
(3) Calculer $B.C$ and $C.B$.
(4) Résoudre le système (S) suivant par deux méthodes :
(méthode d'inversion matricielle et méthode de Cramer)

$$(S) \begin{cases} 2x + y = 3 \\ x + 3y + z = 5 \\ y + 2z = 3 \end{cases}$$

(5) Déterminer les valeurs de $\lambda \in \mathbb{R}$ pour que : $\det(B - \lambda I_3) = 0$.

Bon courage

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Answer key to remedial exam

Exercise 1 (7 pts)

Let $E = Span\{(1,1,1)\}$ and $F = \{(x,y,z) \in \mathbb{R}^3 : x+y-z = 0\}$

(1) Show that F is a subspace of $\mathbb{R}^3$ and find its dimension.

F is a subspace of $\mathbb{R}^3 \Leftrightarrow \begin{cases} (i) & F \neq \varnothing \ (0_{\mathbb{R}^3} \in F) \\ (ii) & \forall u,v \in F : u+v \in F \\ (iii) & \forall u \in F, \forall \alpha \in \mathbb{R} : \alpha u \in F \end{cases}$

(i) $0_{\mathbb{R}^3} = (0,0,0) \in F$, because $0+0-0 = 0$ then $F \neq \varnothing$.

(ii) Let $u = (x_1,y_1,z_1), v = (x_2,y_2,z_2) \in F \Rightarrow x_1+y_1-z_1 = 0$ and $x_2+y_2-z_2 = 0$

$u+v = (x_1,y_1,z_1)+(x_2,y_2,z_2) = (x_1+x_2,y_1+y_2,z_1+z_2) = (X,Y,Z)$

where $X = x_1+x_2,\ Y = y_1+y_2,\ Z = z_1+z_2$

We have $X+Y-Z = (x_1+x_2)+(y_1+y_2)-(z_1+z_2) = (x_1+y_1-z_1)+(x_2+y_2-z_2) = 0+0 = 0$

So, $u+v \in F$

(iii) Let $u = (x,y,z) \in f$ and $\alpha \in \mathbb{R}$.

$u = (x,y,z) \in F \Rightarrow x+y-z = 0$

$\alpha u = \alpha(x,y,z) = (\alpha x, \alpha y, \alpha z)$

We have $(\alpha x)+(\alpha y)-(\alpha z) = \alpha(x+y-z) = \alpha.0 = 0$

Hence $\alpha u \in F$.

Conclusion : F is subspace of $\mathbb{R}^3$

Let $u = (x,y,z) \in F$.

$u = (x,y,z) \in F \Rightarrow x+y-z = 0 \Rightarrow z = x+y$

$u \in F \Rightarrow u = (x,y,x+y) = x(1,0,1)+y(0,1,1) \Rightarrow F = Span = \{(1,0,1),(0,1,1)\}$.

Let $\lambda_1, \lambda_2 \in \mathbb{R}$.

$\lambda_1(1,0,1)+\lambda_2(0,1,1) = (0,0,0) \Rightarrow (\lambda_1,\lambda_2,\lambda_1+\lambda_2) = (0,0,0)$

$$\Rightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \\ \lambda_1+\lambda_2 = 0 \end{cases} \Rightarrow \lambda_1 = \lambda_2 = 0$$

Hence $\{u_1,u_2\}$ is linearly independent and therefore it is a basis of F,
and since $card(\{u_1,u_2\}) = 2$ then $\dim(F) = 2$.

(2) Find $E \cap F$

$u = (x,y,z) \in E \Rightarrow u = \alpha(1,1,1) \Rightarrow x = y = z$

Let $u = (x,y,z) \in E \cap F \Rightarrow \begin{cases} (x,y,z) \in E \\ (x,y,z) \in F \end{cases}$

$$\Rightarrow \begin{cases} x = y = z \\ x+y-z = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}$$

Hence $E \cap F = \{0_{\mathbb{R}^3}\}$

(3) Show that $\mathbb{R}^3 = E \oplus F$.

$$\mathbb{R}^3 = E \oplus F \Leftrightarrow \begin{cases} \dim(E) + \dim(F) = \dim(\mathbb{R}^3) \\ E \cap F = \{0_{\mathbb{R}^3}\} \end{cases}$$

We have $E = Span\{(1,1,1)\}$ and since $(1,1,1) \neq (0,0,0)$, then $\{(1,1,1)\}$ is a basis of E.
Therfore $\dim(E) = 1$.

We have We have $\begin{cases} \dim(E) + \dim(F) = 1 + 2 = 3 = \dim(\mathbb{R}^3) \\ E \cap F = \{0_{\mathbb{R}^3}\} \end{cases} \Leftrightarrow \mathbb{R}^3 = E \oplus F$

Exercise 2 $(\mathbf{5\ pts})$

We consider the map F defined by :

$$F : \mathbb{R}_2[X] \to \mathbb{R}_2[X]$$
$$P \mapsto (X+1)P'$$

(1) Let's show that F is a linear map.
Let $P, Q \in F$ and $\alpha, \beta \in \mathbb{R}$.

$$F(\alpha P + \beta Q) = (X+1)(\alpha P + \beta Q)' = (X+1)(\alpha P' + \beta Q')$$
$$= \alpha(X+1)P' + \beta(X+1)Q'$$
$$F(\alpha P + \beta Q) = \alpha F(P) + \beta F(Q)$$

So, F is a linear map.
(2) Let's determine $\ker(F)$ and deduce $r(F)$, the rank of F.
$\ker(F) = \{P \in \mathbb{R}_2[X] : F(P) = 0\}$
$F(P) = 0 \Rightarrow (X+1)P' = 0 \Rightarrow P' = 0 \Rightarrow P = c \in \mathbb{R}$
So, $\ker(F) = \{P = c : a \in \mathbb{R}\} = Span\{1\}$
Terefore $\{1\}$ is a basis of $\ker(F)$, so $\dim(\ker(F)) = 1$.
By rank theorem, we have : $\dim(\mathbb{R}_2[X]) = \dim(\ker(F)) + r(F)$
then $r(F) = \dim(\mathbb{R}_2[X]) - \dim(\ker(F)) = 3 - 1 = 2 = r(F)$
(3) F is not injective because $\ker(F) = Span\{1\} \neq \{0\}$
We have $r(F) = \dim(\mathrm{Im}(F)) = 2 \neq 3$, which implies that F is not surjective.

Exercise 3 $(\mathbf{8\ pts})$

Let $M = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 5 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 5 \end{pmatrix}$

(1) Let's determine the matrices $A = {}^tM.M$ and $B = A + I_3$

$$M = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \Rightarrow {}^t M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$A = {}^tM.M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} . \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$B = A + I_3 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} + \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

(2) Calculate $\det(B)$.

$$\det(B) = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 2\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + 0\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} = 10 - 2 = 8$$

(3) Let’s calculate $B.C$ and $C.B$.

$$B.C = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix} . \begin{pmatrix} 5 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 5 \end{pmatrix} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix} = 8\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 8I_3$$

$$C.B = \begin{pmatrix} 5 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 5 \end{pmatrix} . \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix} = 8\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 8I_3$$

(4) Let’s solve the following system (S) by two methods :

$$(S)\begin{cases} 2x + y = 3 \\ x + 3y + z = 5 \\ y + 2z = 3 \end{cases}$$

Let’s write the system (S) in matrix form $(DX = b)$.

$(S) \Leftrightarrow DX = b$ with $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $b = \begin{pmatrix} 3 \\ 5 \\ 3 \end{pmatrix}$ and $D = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix} = B$

(i) Let’s solve the system (S) by matrix inverse method

$(S) \Leftrightarrow BX = b$

From (3), we have $B.C = C.B = 8I_3$, which implies that B is invertible and $B^{-1} = \frac{1}{8}C$

$$B^{-1} = \frac{1}{8}\begin{pmatrix} 5 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 5 \end{pmatrix} = \begin{pmatrix} \frac{5}{8} & -\frac{1}{4} & \frac{1}{8} \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{8} & -\frac{1}{4} & \frac{5}{8} \end{pmatrix}$$

$X = B^{-1}b$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{5}{8} & -\frac{1}{4} & -\frac{1}{8} \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{8} & -\frac{1}{4} & \frac{5}{8} \end{pmatrix}\begin{pmatrix} 3 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{15}{8} - \frac{5}{4} + \frac{3}{8} \\ -\frac{3}{4} + \frac{5}{2} - \frac{3}{4} \\ \frac{3}{8} - \frac{5}{4} + \frac{15}{8} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

So, $\begin{cases} x = 1 \\ y = 1 \\ z = 1 \end{cases}$

(ii) Let’s solve the system (S) by Cramer method

$(S) \Leftrightarrow BX = b$

$\det(B) = 8 \neq 0 \Rightarrow (S)$ admits a unique solution given by :

$$x = \frac{\det(B_1)}{\det(B)} = \frac{\begin{vmatrix} \mathbf{3} & 1 & 0 \\ \mathbf{5} & 3 & 1 \\ \mathbf{3} & 1 & 2 \end{vmatrix}}{8} = \frac{1}{8}\left[3\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} - \begin{vmatrix} 5 & 1 \\ 3 & 2 \end{vmatrix} + 0\begin{vmatrix} 5 & 3 \\ 3 & 1 \end{vmatrix}\right] = 1$$

$$y = \frac{\det(B_2)}{\det(B)} = \frac{\begin{vmatrix} 2 & \mathbf{3} & 0 \\ 1 & \mathbf{5} & 1 \\ 0 & \mathbf{3} & 2 \end{vmatrix}}{8} = \frac{1}{8}\left[2\begin{vmatrix} 5 & 1 \\ 3 & 2 \end{vmatrix} - 3\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + 0\begin{vmatrix} 1 & 5 \\ 0 & 3 \end{vmatrix}\right] = 1$$

$$z = \frac{\det(B_3)}{\det(B)} = \frac{\begin{vmatrix} 2 & 1 & \mathbf{3} \\ 1 & 3 & \mathbf{5} \\ 0 & 1 & \mathbf{3} \end{vmatrix}}{8} = \frac{1}{8}\left[2\begin{vmatrix} 3 & 5 \\ 1 & 3 \end{vmatrix} - \begin{vmatrix} 1 & 5 \\ 0 & 3 \end{vmatrix} + 3\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix}\right] = 1$$

So, $\begin{cases} x = 1 \\ y = 1 \\ z = 1 \end{cases}$

(5) Determine the values of $\lambda \in \mathbb{R}$ so that : $\det(B - \lambda I_3) = 0$.

$$B - \lambda I_3 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2-\lambda & 1 & 0 \\ 1 & 3-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{pmatrix}$$

$$\det(B - \lambda I_3) = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 3-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 0 & 2-\lambda \end{vmatrix} + 0\begin{vmatrix} 1 & 3-\lambda \\ 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)[(3-\lambda)(2-\lambda) - 1] - (2-\lambda) = 0$$

$$\Rightarrow (2-\lambda)[(3-\lambda)(2-\lambda) - 2] = 0 \Rightarrow (2-\lambda)(\lambda^2 - 5\lambda + 4) = 0$$

$$\Rightarrow \begin{cases} (2-\lambda) = 0 \\ \lambda^2 - 5\lambda + 4 = 0 \end{cases} \Rightarrow \begin{cases} \lambda = 2 \\ \lambda = 1 \text{ ou } \lambda = 4 \end{cases}$$

So, $\det(B - \lambda I_3) = 0 \Leftrightarrow \lambda \in \{1,2,4\}$.