Université Abou Bekr Belkaid Tlemcen. $1^{ère}$ année LMD (2023/2024).
Faculté des Sciences S2, durée 1h:30'.

Examen Final -Analyse 2-M et MI.

Question de cours **(05 points)** **Intégrer :** $I = \int \frac{dx}{(1+x^2)^3}$.

Exercice **(15 points)** Les questions suivantes sont complètement indépendantes.

1. **(5 points).** Soit la fonction définie par $f(x) = (\arctan x)\, e^{\frac{1}{x}}$.
   Trouver l'équation de **l'asymptote** au graphe de f dans $V(+\infty)$, ainsi que sa **position** par rapport à ce graphe.

2. **(5 points).** **Calculer** les intégrales suivantes :

$$a)\ \ I = \int \sin^8(x)\cos^3 x dx \quad ; \quad b)\ \ J = \int \frac{\sin x\ \cos x}{(\sin^2 x - 1)(\sin^2 x + \sin x + 1)} dx$$

3. **(5 points).** **Résoudre** l'EDO suivante :

$$xy'' + y' + x = 0; (x > 0)$$

On rappelle que :
1/ $\forall \alpha \in \mathbb{R}$, $\cos^2\alpha = \frac{1}{2}(\cos(2\alpha)+1)$ et $\cos^2\alpha = \dfrac{1}{1+\tan^2\alpha}$.
2/ $\forall y > 0$, $\arctan\left(\dfrac{1}{y}\right) = \dfrac{\pi}{2} - \arctan y$.
3/ Au voisinage de 0, on a :
a) $\arctan(x) = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} ... + \frac{(-1)^n x^{2n+1}}{(2n+1)} + o(x^{2n+1})$.
b) $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + ... + \dfrac{x^n}{n!} + o(x^n)$.

Remarque: L'usage des téléphones portables est strictement interdit. Bon courage.

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Final Exam - Calculus 2 - M and MI.

Theoretical question **(05 points)** **Integrate :** $I = \int \frac{dx}{(1+x^2)^3}$.

Exercise **(15 points)** The following questions are completely independent.

1. **(5 points).** Let be the function defined by $f(x) = (\arctan x)\, e^{\frac{1}{x}}$.
   Find the **asymptote** equation to the plot of the function f in $V(+\infty)$, as well as its **position** with respect to this plot.

2. **(5 points).** **Calculate** the following integrals :

$$a)\;\; I = \int \sin^8(x)\cos^3 x dx \quad ; \quad b)\;\; J = \int \frac{\sin x \;\; \cos x}{(\sin^2 x - 1)(\sin^2 x + \sin x + 1)} dx$$

3. **(5 points).** **Solve** the following ODE:

$$xy'' + y' + x = 0; (x > 0)$$

Let's recall that :
1/ $\forall \alpha \in \mathbb{R}$, $\cos^2\alpha = \frac{1}{2}(\cos(2\alpha)+1)$ and $\cos^2\alpha = \dfrac{1}{1+\tan^2\alpha}$.
2/ $\forall y > 0$, $\arctan\left(\dfrac{1}{y}\right) = \dfrac{\pi}{2} - \arctan y$.
3/ In the neighborhood of 0, one has:
a) $\arctan(x) = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} ... + \frac{(-1)^n x^{2n+1}}{(2n+1)} + o(x^{2n+1})$.
b) $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + ... + \dfrac{x^n}{n!} + o(x^n)$.

Note: The use of cell phones is strictly prohibited. Good luck.

Final Exam Correction- Calculus 2 - M and MI.

Theoretical question (05 points) Integrate : $K = \int \frac{dx}{(1+x^2)^3}$.

Take $x = \tan y$, to get : $dx = (1+\tan^2 y)dy$, so : **(1+1pt)**

$$\begin{aligned} K &= \int \frac{(1+\tan^2 y)}{(1+\tan^2 y)^3} dy \\ &= \int \frac{1}{(1+\tan^2 y)^2} dy \\ \text{But since } \forall y &\in \mathbb{R}, \cos^2 y = \frac{1}{(1+\tan^2 y)}, \text{ one gets :} \\ K &= \int \cos^4 y \; dy \quad \textbf{(0.5pt)} \end{aligned}$$

We know how to calculate $\int \cos^4 y \; dy$ by linearization since: $\forall y \in \mathbb{R}, \cos^2 y = \frac{1}{2}(1+\cos(2y))$, so:

$$\begin{aligned} \int \cos^4 y \; dy &= \frac{1}{4}\int (1+\cos(2y))^2 dy \; \textbf{(0.5pt)} \\ &= \frac{1}{4}\int (1+2\cos(2y)+\cos^2(2y))dy \\ &= \frac{1}{4}\left[y + \sin(2y) + \frac{1}{2}\int (1+\cos(4y))dy \right] \; \textbf{(0.25+0.25 pt)} \\ &= \frac{1}{4}\left[y + \sin(2y) + \frac{1}{2}(y + \frac{1}{4}\sin(4y)\right] + C, (C \in \mathbb{R}) . \; \textbf{(0.5pt)} \end{aligned}$$

It remains to return to the first variable, noting that :

$$x = \tan y \Rightarrow y = \arctan x, \text{ so } \sin(2y) = \sin(2\arctan x) \text{ and } \sin(4y) = \sin(4\arctan x) \quad \textbf{(0.5pt)}$$

Finally :

$$K = \frac{1}{4}\left[\frac{3}{2}\arctan x + \sin(2\arctan x) + \frac{1}{8}\sin(4\arctan x) \right] + C, (C \in \mathbb{R}) . \; \textbf{(0.5pt)}$$

Exercise (15 points)

1. Let be the function defined by $f(x) = e^{\frac{1}{x}}(\arctan x)$.

   Find the asymptote equation to the plot of the function f in $V(+\infty)$, as well as its position with respect to this plot.

   One poses $y = \frac{1}{x}$, that way, $\lim_{x\to+\infty} y = 0^+$ **(0.5+0.5pt)** and $f\left(\frac{1}{y}\right) = e^y \arctan\left(\frac{1}{y}\right)$.

   Since $y > 0$, one has : $\arctan\left(\frac{1}{y}\right) = \frac{\pi}{2} - \arctan y$, so $\arctan\left(\frac{1}{y}\right) = \frac{\pi}{2} - (y + o(y^2))$. **(1pt)**

And $e^y = 1 + y + \dfrac{y^2}{2} + o(y^2)$ so,

$$\begin{aligned} f\left(\frac{1}{y}\right) &= \left[1 + y + \frac{y^2}{2} + o(y^2)\right]\left[\frac{\pi}{2} - y + o(y^2)\right] \\ &= \frac{\pi}{2} - y + \frac{\pi}{2}y - y^2 + \frac{\pi}{4}y^2 + o(y^2) \\ &= \frac{\pi}{2} + \left(\frac{\pi}{2} - 1\right)y + \left(\frac{\pi}{4} - 1\right)y^2 + o(y^2) \end{aligned}$$

$$\Rightarrow f(x) = \frac{\pi}{2} + \left(\frac{\pi}{2} - 1\right)\frac{1}{x} + \left(\frac{\pi}{4} - 1\right)\left(\frac{1}{x}\right)^2 + o\left(\left(\frac{1}{x}\right)^2\right). \textbf{(1pt)}$$

This means that the asymptote equation to the plot of the function f in $V(+\infty)$ is the line (Δ) of equation $y = \dfrac{\pi}{2}$. It is an horizontal asymptote. **(1pt)**

And since $\forall x \in V(+\infty), (f(x) - y) = \left(\dfrac{\pi}{2} - 1\right)\dfrac{1}{x} > 0$ then the plot of f is **above** (Δ). **(1pt).**

2. **integrate**

a)

$$I = \int \sin^8(x)\cos^3 x dx$$

One has:

$$\begin{aligned} \forall x \in \mathbb{R}, \quad \sin^8(x)\cos^3(x) &= \sin^8(x)\cos^2(x)\cos(x) \quad \textbf{(0.5pt)} \\ &= \sin^8(x)\left[1 - \sin^2(x)\right]\cos(x) \quad \textbf{(0.5 pt)} \\ &= \sin^8(x)\cos x - \sin^{10}(x)\cos(x) \end{aligned}$$

Hence :

$$\begin{aligned} I &= \int \sin^8(x)\cos x\ dx - \int \sin^{10}(x)\cos(x)\ dx \\ &= \frac{\sin^9(x)}{9} - \frac{\sin^{11}(x)}{11} + C, (C \in \mathbb{R}). \quad \textbf{(0.5+0.5 pt)} \end{aligned}$$

b)

$$J = \int \frac{\sin x\ \cos x}{(\sin^2 x - 1)(\sin^2 x + \sin x + 1)}dx$$

One poses $y = \sin x$, to get $dy = \cos x dx$, **(0.5 pt)** so :

$$J = \int \frac{ydy}{(y^2 - 1)(y^2 + y + 1)} = \int f(y)dy$$

One has :

$$\begin{aligned} f(y) &= \frac{y}{(y^2 - 1)(y^2 + y + 1)} = \frac{A}{y - 1} + \frac{B}{y + 1} + \frac{Cy + D}{y^2 + y + 1} \textbf{(0.5+0.5+0.5 pt)} \qquad ((*)) \\ D_f &= \mathbb{R}/\{-1, 1\} \end{aligned}$$

We multiply the whole equation $(*)$ by $(y-1)$, we find :

$$\forall y \in \mathbb{R}/\{-1\}, \frac{y}{(y+1)(y^2+y+1)} = A + \frac{B(y-1)}{y+1} + \frac{(Cy+D)(y-1)}{y^2+y+1}.$$

Take $y=1$ to get :

$$A = \frac{1}{6} \quad \textbf{(0.5pt)}$$

Similarly, we multiply the whole equation $(*)$ by $(y+1)$, we find :

$$\forall y \in \mathbb{R}/\{1\}, \frac{y}{(y-1)(y^2+y+1)} = \frac{A(y+1)}{y-1} + B + \frac{(Cy+D)(y+1)}{y^2+y+1}$$

We now choose to take $y=-1$, we find directly :

$$B = \frac{1}{2} \quad \textbf{(0.5pt)}$$

Now we come back to our equation given in $(*)$:

$$\forall y \in D_f, \frac{y}{(y^2-1)(y^2+y+1)} = \frac{A}{y-1} + \frac{B}{y+1} + \frac{Cy+D}{y^2+y+1}$$

Since we have already calculated A and B, we may take directly $y=0$ to find the D, as follows :

$$\begin{aligned} 0 &= -A+B+D \\ &\Rightarrow 0 = -\frac{1}{6} + \frac{1}{2} + D \\ &\Rightarrow D = -\frac{1}{3} \quad \textbf{(0.5pt)} \end{aligned}$$

And finally, we multiply the whole equation $(*)$ by y then pass to the limit when y tends to $+\infty$:

$$\begin{aligned} \forall y \in D_f, & \frac{y^2}{(y^2-1)(y^2+y+1)} = \frac{Ay}{y-1} + \frac{By}{y+1} + \frac{(Cy+D)y}{y^2+y+1} \\ \Rightarrow & \lim_{y\to+\infty} \frac{y^2}{(y^2-1)(y^2+y+1)} = \lim_{y\to+\infty} \left[\frac{Ay}{y-1} + \frac{By}{y+1} + \frac{Cy^2+Dy}{y^2+y+1}\right] \\ \Rightarrow & \ 0 = A+B+C \\ \Rightarrow & \ 0 = \frac{1}{6} + \frac{1}{2} + C \\ \Rightarrow & \ C = -\frac{2}{3}.\textbf{(0.5pt)} \end{aligned}$$

Hence :

$$\forall y \in D_f, \frac{y}{(y^2-1)(y^2+y+1)} = \frac{\frac{1}{6}}{y-1} + \frac{\frac{1}{2}}{y+1} + \frac{-\frac{2}{3}y - \frac{1}{3}}{y^2+y+1}$$

All that remains is to integrate:

$$I = \int \frac{y\,dy}{(y^2-1)(y^2+y+1)} = \frac{1}{6}\int \frac{dy}{y-1} + \frac{1}{2}\int \frac{dy}{y+1} - \frac{1}{3}\int \frac{2y+1}{y^2+y+1}dy$$

$$I = \frac{1}{6}\ln|y-1| + \frac{1}{2}\ln|y+1| - \frac{1}{3}\left[\ln\left|y^2+y+1\right|\right] + C, (C \in \mathbb{R}).\textbf{(0.25+0.25+0.25pt)}$$

Finally,

$$I = \frac{1}{6}\ln|\sin x - 1| + \frac{1}{2}\ln|\sin x + 1| - \frac{1}{3}\ln\left(\sin^2 x + \sin x + 1\right) + C, (C \in \mathbb{R}) . \mathbf{(0.25pt)}$$

3. Solve the following ODE:

$$xy'' + y' + x = 0; (x > 0) ... (1)$$

One poses $z = y'$, one finds $z' = y''$ (**0.5+0.5 pt)** so that equation (1) becomes:

$$(1) \Leftrightarrow xz' + z + x = 0 ... (2)$$

Which is a linear first order ODE:

$$(2) \Leftrightarrow z' + \frac{1}{x}z = -1 \text{ (remember that } x > 0) \quad \mathbf{(1pt)}$$

With $a(x) = \frac{1}{x}$ and $b(x) = -1$.
Let's solve it by the integrating factor method :
$\int a(x)dx = \ln x$ so the integrating factor is $R(x) = e^{\int a(x)dx} = e^{\ln x} = x$.**(1pt)**
Furthermore, $I(x) = \int R(x)b(x)dx = \int -xdx = \frac{-x^2}{2}$.
The solution is then given by :

$$\begin{aligned} R(x)z(x) &= (I(x) + C_1) . \\ &\Longrightarrow z(x) = \frac{1}{x}\left(\frac{-x^2}{2} + C_1\right) \\ &\Rightarrow z(x) = \frac{-x}{2} + \frac{C_1}{x} \quad \mathbf{(1pt)} \end{aligned}$$

And since $y(x) = \int z(x)dx$ then $y(x) = \int \left(\frac{-x}{2} + \frac{C_1}{x}\right) dx$

$$\Rightarrow y(x) = \frac{-x^2}{4} + C_1 \ln x + C_2. \quad (C_1, C_2 \in \mathbb{R}), (x > 0). \quad \mathbf{(1pt)}$$