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Academic year: 2023 - 2024
Module : Algebra 2
L1 Math+MI

Final exam

Exercise 1 $(\mathbf{10\ pts})$

Let $F = \{(x,y,z) \in \mathbb{R}^3 : -x+y-z=0\}$ and f be a map defined by :

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$(x,y,z) \mapsto \big(x+y,\ y-z,\ z+x\big)$$

(1) We admit that F is a subspace of $\mathbb{R}^3$. Find its dimension.
(2) Prove that f is a linear map.
(3) Show that $\ker(f) = \{(x,y,z) \in \mathbb{R}^3 : y+x=0 \text{ and } z+x=0\}$.
(4) Determine $\dim(\ker(f))$ and deduce $r(f)$, the rank of f.
(5) f is it injective? Surjective? Justify your answer.
(6) Do we have $\mathbb{R}^3 = \ker(f) \oplus F$.

Exercise 2 $(\mathbf{10\ pts})$

We consider the following system (S) :

$$(S)\begin{cases} x-3y+6z=3 \\ 6x-8y+12z=2 \\ 3x-3y+4z=1 \end{cases}$$

(1) Write the system (S) in matrix form $(AX=B)$.
(2) Check that the matrix A is invertible.
(3) Calculate A^2.
(4) Determine the real numbers α and β such that :

$$A^2 = \alpha A + \beta I_3$$

where I_3 is the unit matrix (identity matrix) of order 3.
(5) Give A^{-1} the inverse matrix of A.
(6) Solve the system (S).

Good luck

Université Abou Bekr Belkaid - Tlemcen
Faculté des sciences
Département de mathématiques

Année universitaire: 2023 - 2024
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Epreuve Finale

Exercice 1 (10 pts)

Soient $F = \{(x,y,z) \in \mathbb{R}^3 : -x + y - z = 0\}$ et f une application définie par :

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$(x,y,z) \mapsto \left(x + y,\ y - z,\ z + x\right)$$

(1) On admet que F est un sous espace vectoriel de $\mathbb{R}^3$. Déterminer sa dimension.
(2) Prouver que f est une application linéaire.
(3) Montrer que $\ker(f) = \left\{(x,y,z) \in \mathbb{R}^3 : y + x = 0 \text{ et } z + x = 0\right\}$.
(4) Déterminer $\dim(\ker(f))$ et en déduire $rg(f)$, le rang de f.
(5) f est-elle injective? Surjective? Justifier votre réponse.
(6) A-t-on $\mathbb{R}^3 = \ker(f) \oplus F$.

Exercice 2 (10 pts)

On considère le système (S) suivant :

$$(S) \begin{cases} x - 3y + 6z = 3 \\ 6x - 8y + 12z = 2 \\ 3x - 3y + 4z = 1 \end{cases}$$

(1) Écrire le système (S) sous la forme matricielle $(AX = B)$.
(2) Vérifier que la matrice A est inversible.
(3) Calculer A^2.
(4) Déterminer les réels α et β tels que :

$$A^2 = \alpha A + \beta I_3$$

où I_3 est la matrice unité d'ordre 3.
(5) Donner A^{-1} la matrice inverse de A.
(6) Résoudre le système (S).

Bon courage

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Answer key to final exam

Exercise 1 $\left(\mathbf{10\ pts}\right)$

(1) Let $u = (x,y,z) \in F$, then $z = -x + y$

$u \in F \Rightarrow u = (x,y,-x+y) = x(1,0,-1) + y(0,1,1) = xu_1 + yu_2$

where $u_1 = (1,0,-1)$ and $u_2 = (0,1,1)$

We deduce that $F = Span\{u_1,u_2\}$, which means : (1)

$C_1 = \{u_1,u_2\}$ is a spanning part of F.

Is C_1 linearly independent?

Let $\lambda_1,\lambda_2 \in \mathbb{R}$.

$\lambda_1 u_1 + \lambda_2 u_2 = 0 \Rightarrow \lambda_1(1,0,-1) + \lambda_2(0,1,1) = (\lambda_1,\lambda_2,-\lambda_1+\lambda_2) = (0,0,0)$

$$\Rightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \\ -\lambda_1 + \lambda_2 = 0 \end{cases} \Rightarrow \lambda_1 = \lambda_2 = 0$$

Hence $C_1 = \{u_1,u_2\}$ is linearly independent and therefore it is a basis of F, (1)
and since $card(C_1) = 2$ then $\dim(F) = 2$.

(2) Let $u = (x_1,y_1,z_1), v = (x_2,y_2,z_2) \in \mathbb{R}^3$. we have

$$\begin{aligned} f(u+v) &= f((x_1,y_1,z_1)+(x_2,y_2,z_2)) = f(x_1+x_2,y_1+y_2,z_1+z_2) = f(X,Y,Z) \\ &= (X+Y, Y-Z, Z+X) \\ &= (x_1+x_2+y_1+y_2, y_1+y_2-z_1-z_2, z_1+z_2+x_1+x_2) \\ &= ((x_1+y_1)+(x_2+y_2), (y_1-z_1)+(y_2-z_2), (z_1+x_1)+(z_2+x_2)) \\ &= (x_1+y_1, y_1-z_1, z_1+x_1) + (x_2+y_2, y_2-z_2, z_2+x_2) \\ &= f(x_1,y_1,z_1) + f(x_2,y_2,z_2) \end{aligned}$$

$f(u+v) = f(u) + f(v)$

Let $u = (x,y,z) \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$. we have (1)

$$\begin{aligned} f(\alpha u) &= f(\alpha(x,y,z)) = f(\alpha x, \alpha y, \alpha z) = f(X,Y,Z) \\ &= (\alpha x + \alpha y, \alpha y - \alpha z, \alpha z + \alpha x) \\ &= \alpha(x + \alpha y, \alpha y - \alpha z, \alpha z + \alpha x) \\ &= \alpha f(x,y,z) \end{aligned}$$

$f(\alpha u) = \alpha f(u)$

Therefore f is a linear map.

(3) $\ker(f) = \{(x,y,z) \in \mathbb{R}^3 : f(x,y,z) = (0,0,0)\}$

Asume that $v = (x,y,z) \in \ker(f)$

$$\begin{aligned} v = (x,y,z) \in \ker(f) &\Rightarrow f(x,y,z) = (0,0,0) \\ &\Rightarrow \left(x+y,\ y-z,\ z+x\right) = (0,0,0) \\ &\Rightarrow \begin{cases} y+x = 0 \\ y-z = 0 \\ z+x = 0 \end{cases} \Rightarrow \begin{cases} y = -x \\ y = z \\ z = -x \end{cases} \Rightarrow \begin{cases} y+x = 0 \\ z+x = 0 \end{cases} \end{aligned}$$ (1)

Hence $\ker(f) = \left\{(x,y,z) \in \mathbb{R}^3 : y+x = 0 \text{ and } z+x = 0\right\}$

(4) Let $v = (x,y,z) \in \ker(f) \Rightarrow y+x = 0$ and $z+x = 0 \Rightarrow y = -x$ and $z = -x$
$v \in \ker(f) \Rightarrow v = (x,-x,-x) = x(1,-1,-1) = xu_3$ where $u_3 = (1,-1,-1)$
Hence, $\ker(f) = Span\{u_3\} \Rightarrow C_2 = \{u_3\}$ is a spanning part of $\ker(f)$.
Since $u_3 \neq 0_{\mathbb{R}^3}$ then $C_2 = \{u_3\}$ is a basis of $\ker(f)$.
And consequently $\dim(\ker(f)) = 1$. (1)
By rank theorem, we have $\dim(\mathbb{R}^3) = \dim(\ker(f)) + r(f)$
then $r(f) = \dim(\mathbb{R}^3) - \dim(\ker(f)) = 3 - 1 = 2$ (1)
The rank of f, $r(f) = 2$
(5) f is not injective because $\ker(f) = Span\{u_3\} \neq \{0_{\mathbb{R}^3}\}$ (1)
$\mathrm{Im}(f) \subset \mathbb{R}^3$ and $\dim(\mathrm{Im}(f)) = r(f) = 2 \neq 3$, so $\mathrm{Im}(f) \neq \mathbb{R}^3$ (1)
Then f is not surjective.

$$(6)\ \mathbb{R}^3 = \ker(f) \oplus F \Leftrightarrow \begin{cases} \dim(\ker(f)) + \dim(F) = \dim(\mathbb{R}^3) \\ \ker(f) \cap F = \{0_{\mathbb{R}^3}\} \end{cases}$$ (1)

We have $\dim(\ker(f)) + \dim(F) = 1 + 2 = 3 = \dim(\mathbb{R}^3)$
So, to show that $\mathbb{R}^3 = \ker(f) \oplus F$, it suffices to show that $\ker(f) \cap F = \{0_{\mathbb{R}^3}\}$.

$$\text{Let } u = (x,y,z) \in \ker(f) \cap F \Rightarrow \begin{cases} (x,y,z) \in \ker(f) \\ (x,y,z) \in F \end{cases}$$

$$\Rightarrow \begin{cases} y+x = 0 \\ z+x = 0 \\ -x+y-z = 0 \end{cases} \Rightarrow \begin{cases} y = -x \\ z = -x \\ -x+y-z = 0 \end{cases}$$ (1)

$$\Rightarrow x = y = z = 0$$

Hence $\ker(f) \cap F = \{0_{\mathbb{R}^3}\}$.

$$\text{We have } \begin{cases} \dim(\ker(f)) + \dim(F) = \dim(\mathbb{R}^3) \\ \ker(f) \cap F = \{0_{\mathbb{R}^3}\} \end{cases} \Leftrightarrow \mathbb{R}^3 = \ker(f) \oplus F.$$

Exercise 2 (**10 pts**)

We consider the following system (S) :

$$(S) \begin{cases} x - 3y + 6z = 3 \\ 6x - 8y + 12z = 2 \\ 3x - 3y + 4z = 1 \end{cases}$$

(1) Let's write the system (S) in matrix form $(AX = B)$.

$$(S) \Leftrightarrow AX = B \text{ with } X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, A = \begin{pmatrix} 1 & -3 & 6 \\ 6 & -8 & 12 \\ 3 & -3 & 4 \end{pmatrix}$$ (1)

$$(2)\ \det(A) = \begin{vmatrix} 1 & -3 & 6 \\ 6 & -8 & 12 \\ 3 & -3 & 4 \end{vmatrix} = \begin{vmatrix} -8 & 12 \\ -3 & 4 \end{vmatrix} - (-3)\begin{vmatrix} 6 & 12 \\ 3 & 4 \end{vmatrix} + 6\begin{vmatrix} 6 & -8 \\ 3 & -3 = \end{vmatrix} = 4$$

$\det(A) = 4 \neq 0 \Rightarrow A$ is invertible. (1,5)

(3) $A^2 = A.A = \begin{pmatrix} 1 & -3 & 6 \\ 6 & -8 & 12 \\ 3 & -3 & 4 \end{pmatrix}\begin{pmatrix} 1 & -3 & 6 \\ 6 & -8 & 12 \\ 3 & -3 & 4 \end{pmatrix}$

$$= \begin{pmatrix} 1-18+18 & -3+24-18 & 6-36+24 \\ 6-48+36 & -18+64-36 & 36-96+48 \\ 3-18+12 & -9+24+12 & 18-36+16 \end{pmatrix}$$ (1)

$$A^2 = \begin{pmatrix} 1 & 3 & -6 \\ -6 & 10 & -12 \\ -3 & 3 & -2 \end{pmatrix}$$

(4) Let's determine the real numbers α and β such that :

$$A^2 = \alpha A + \beta I_3$$

$$A^2 = \alpha A + \beta I_3 = \alpha\begin{pmatrix} 1 & -3 & 6 \\ 6 & -8 & 12 \\ 3 & -3 & 4 \end{pmatrix} + \beta\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & -6 \\ -6 & 10 & -12 \\ -3 & 3 & -2 \end{pmatrix} = \begin{pmatrix} \alpha+\beta & -3\alpha & 6\alpha \\ 6\alpha & -8\alpha+\beta & 12\alpha \\ 3\alpha & -3\alpha & 4\alpha+\beta \end{pmatrix} \Rightarrow \begin{cases} \alpha = -1 \\ \beta = 2 \end{cases}$$ (1,5)

So, $A^2 = -A + 2I_3$

(5) $A^2 = -A + 2I_3 \Rightarrow A^2 + A = 2I_3 \Rightarrow A(A + I_3) = 2I_3$

$$\Rightarrow A\left[\frac{1}{2}(A + I_3)\right] = I_3$$ (1)

Hence, A is invertible and $A^{-1} = \frac{1}{2}(A + I_3)$

$$A^{-1} = \frac{1}{2}\left(\begin{pmatrix} 1 & -3 & 6 \\ 6 & -8 & 12 \\ 3 & -3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 1+1 & -3+0 & 6+0 \\ 6+0 & -8+1 & 12+0 \\ 3+0 & -3+0 & 4+1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1 & -\frac{3}{2} & 3 \\ 3 & -\frac{7}{2} & 6 \\ \frac{3}{2} & -\frac{3}{2} & \frac{5}{2} \end{pmatrix}$$ (1)

(6) Let's solve the system (S).

(i) Matrix inverse method

$(S) \Leftrightarrow AX = B \Leftrightarrow A^{-1}AX = A^{-1}B \Leftrightarrow X = A^{-1}B$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -\frac{3}{2} & 3 \\ 3 & -\frac{7}{2} & 6 \\ \frac{3}{2} & -\frac{3}{2} & \frac{5}{2} \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1.3 + (-\frac{3}{2}).2 + 3.1 \\ 3.3 + (-\frac{7}{2}).2 + 6.1 \\ \frac{3}{2}.3 + (-\frac{3}{2}).2 + \frac{5}{2}.1 \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \\ 4 \end{pmatrix}$$ (2)

So, $\begin{cases} x = 3 \\ y = 8 \\ z = 4 \end{cases}$

(ii) Second method : Cramer method

$(S) \Leftrightarrow AX = B$

$\det(A) = 4 \neq 0 \Rightarrow (S)$ admits a unique solution given by :

$$x = \frac{\det(A_1)}{\det(A)} = \frac{\begin{vmatrix} \mathbf{3} & -3 & 6 \\ \mathbf{2} & -8 & 12 \\ \mathbf{1} & -3 & 4 \end{vmatrix}}{4} = \frac{1}{4}\left[3\begin{vmatrix} -8 & 12 \\ -3 & 4 \end{vmatrix} - (-3)\begin{vmatrix} 2 & 12 \\ 1 & 4 \end{vmatrix} + 6\begin{vmatrix} 2 & -8 \\ 1 & -3 \end{vmatrix}\right] = 3$$

(2)

$$y = \frac{\det(A_2)}{\det(A)} = \frac{\begin{vmatrix} 1 & \mathbf{3} & 6 \\ 6 & \mathbf{2} & 12 \\ 3 & \mathbf{1} & 4 \end{vmatrix}}{4} = \frac{1}{4}\left[\begin{vmatrix} 2 & 12 \\ 1 & 4 \end{vmatrix} - 3\begin{vmatrix} 6 & 12 \\ 3 & 4 \end{vmatrix} + 6\begin{vmatrix} 6 & 2 \\ 3 & 1 \end{vmatrix}\right] = 8$$

$$z = \frac{\det(A_2)}{\det(A)} = \frac{\begin{vmatrix} 1 & -3 & \mathbf{3} \\ 6 & -8 & \mathbf{2} \\ 3 & -3 & \mathbf{1} \end{vmatrix}}{4} = \frac{1}{4}\left[\begin{vmatrix} -8 & 2 \\ -3 & 1 \end{vmatrix} - (-3)\begin{vmatrix} 6 & 2 \\ 3 & 1 \end{vmatrix} + 3\begin{vmatrix} 6 & -8 \\ 3 & -3 \end{vmatrix}\right] = 4$$

So, $\begin{cases} x = 3 \\ y = 8 \\ z = 4 \end{cases}$