Abou Bekr Belkaid University of Tlemcen. 1st grade LMD (2023/2024).

Faculty of Sciences, Mathematics departement M-MI,

First and last name :

Test 1 : Elementary functions, G2

1. Calculate : $\arccos\left(\cos\frac{-6\pi}{4}\right)$. $(1point)$.
2. Show that $\forall x \in [-1,1], \arccos x + \arcsin x = \frac{\pi}{2}$. $(5points)$.

Abou Bekr Belkaid University of Tlemcen. 1st grade LMD (2023/2024).

Faculty of Sciences, Mathematics departement M-MI,

First and last name :

Test 1 : Elementary functions, G3 and G6

1. Calculate: $\arcsin\left(\sin\frac{7\pi}{3}\right)$. $(1 point)$.

2. Show that, for $x \in \mathbb{R}^*, \arctan x + \arctan\left(\frac{1}{x}\right) = \left\{\begin{array}{l} \frac{\pi}{2} \\ -\frac{\pi}{2} \end{array}\right.$. $(5 points)$.

Abou Bekr Belkaid University of Tlemcen. 1st grade LMD (2023/2024).

Faculty of Sciences, Mathematics departement M-MI,

First and last name :

Test 1 : Elementary functions, G1

1. Calculate: $\arcsin\left(\sin\frac{\pi}{6}\right)$. $(1point)$.
2. Show that $\forall \alpha \geq 1,\ \arg ch(\alpha) = \ln(\alpha + \sqrt{\alpha^2 - 1})$. $(5points)$.

Abou Bekr Belkaid University of Tlemcen. 1st grade LMD (2023/2024).

Faculty of Sciences, Mathematics departement M-MI,

First and last name :

Test 1 : Elementary functions, G4 and G5

1. Calculate : $\arccos\left(\cos\frac{2\pi}{3}\right)$. $(1 point)$.
2. Show that $\forall\alpha\in\mathbb{R},\ \arg sh(\alpha)=\ln(\alpha+\sqrt{\alpha^2+1})$. $(5 points)$.

Test 1 correction : Elementary functions, G2

1. Calculate : $\arccos\left(\cos\frac{-6\pi}{4}\right)$.
   $\frac{-6\pi}{4}=\frac{-3\pi}{2}$ so $\cos\frac{-6\pi}{4}=\cos\frac{-3\pi}{2}=\cos\frac{\pi}{2}$, which gives :

$$\arccos\left(\cos\frac{-6\pi}{4}\right)=\arccos\left(\cos\frac{\pi}{2}\right)=\frac{\pi}{2}.\quad (1point).$$

2. Show that $\forall x\in[-1,1], \arccos x+\arcsin x=\frac{\pi}{2}$.

- **1st way:** Put : $\forall x\in[-1,1], f(x)=\arccos x+\arcsin x.\quad (1point).$

f is countinuous over $[-1,1]$ and derivable over $]-1,1[$ as follows :

$$f'(x)=\frac{-1}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1-x^2}}=0\quad (1point).$$

So f is constant on $]-1,1[$. i.e.:$\forall x\in\,]-1,1[\,, f(x)=f(0).\quad (1point).$

$$\begin{aligned} f(0) &= \arccos(0)+\arcsin(0)\\ &= \frac{\pi}{2}+0=\frac{\pi}{2}.\quad (0.5point). \end{aligned}$$

For $x=1$:

$$\begin{aligned} f(1) &= \arccos(1)+\arcsin(1)\\ &= 0+\frac{\pi}{2}=\frac{\pi}{2}.\quad (0.5point). \end{aligned}$$

For $x=-1$:

$$\begin{aligned} f(-1) &= \arccos(-1)+\arcsin(-1)\\ &= \pi-\frac{\pi}{2}=\frac{\pi}{2}.\quad (0.5point). \end{aligned}$$

Finally:

$$\boxed{\forall x\in[-1,1],\ f(x)=\frac{\pi}{2}}\quad (0.5point).$$

- **2nd way** One has :

$$\begin{aligned} x &\in [-1,1]\Rightarrow \arccos x\in[0,\pi]\quad (1point).\\ &\Rightarrow \frac{\pi}{2}-\arccos x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\quad (1point). \end{aligned}$$

Now, we may apply sin:

$$\sin\left(\frac{\pi}{2}-\arccos x\right)=\cos(\arccos x)\quad (1point).$$

Because we have the trigonometric relationship:

$$\forall\alpha\in\mathbb{R}, \sin\left(\frac{\pi}{2}-\alpha\right)=\cos\alpha$$

So :

$$\begin{aligned} \sin\left(\frac{\pi}{2}-\arccos x\right) &= x\quad (1point).\\ &\Rightarrow \frac{\pi}{2}-\arccos x=\arcsin x \end{aligned}$$

$$\boxed{\Rightarrow x\in[-1,1],\ \frac{\pi}{2}=\arccos x+\arcsin x}\quad (1point).$$

Test 1 correction: Elementary functions, G3 and G6

1. Calculate: $\arcsin\left(\sin\frac{7\pi}{3}\right)$.

$\frac{7\pi}{3} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we need to simplify it :

$$\frac{7\pi}{3} = \frac{6\pi+\pi}{3} = 2\pi + \frac{\pi}{3}$$

So,

$$\sin\frac{7\pi}{3} = \sin\frac{\pi}{3},$$

Which means that :

$$\arcsin\left(\sin\frac{7\pi}{3}\right) = \arcsin\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3}. \quad (1point).$$

2. Show that, for $x \in \mathbb{R}^*, \arctan x + \arctan\left(\frac{1}{x}\right) = \begin{cases} \frac{\pi}{2} \\ -\frac{\pi}{2} \end{cases}$.

- **1st way :** Put : $\forall x \in \mathbb{R}^*,\ f(x) = \arctan x + \arctan\left(\frac{1}{x}\right)$ $(1point)$.

f is continuous and derivable over $\mathbb{R}^*$ as follows :

$$\begin{aligned} f'(x) &= \frac{1}{1+x^2} + \frac{-\frac{1}{x}}{1+\left(\frac{1}{x}\right)^2} \\ &= \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0 \quad (1point). \end{aligned}$$

As there is a discontinuity at point 0, f est constant $(0.5point)$ over $]-\infty, 0[\cup]0, +\infty[$ in the following way :

$$f(x) = \begin{cases} k_1 & \text{if } x > 0 \\ k_2 & \text{if } x < 0 \end{cases} \quad (1point).$$

For $x = 1$:

$$\begin{aligned} k_1 &= f(1) \\ &= \arctan 1 + \arctan 1 \\ &= \frac{\pi}{4} + \frac{\pi}{4} \\ &\Rightarrow k_1 = \frac{\pi}{2}. \quad (0.5point). \end{aligned}$$

For $x = -1$:

$$\begin{aligned} k_2 &= f(-1) \\ &= \arctan(-1) + \arctan(-1) \\ &= -\frac{\pi}{4} - \frac{\pi}{4} \\ &\Rightarrow k_2 = -\frac{\pi}{2}. \quad (0.5point). \end{aligned}$$

Finally :

$$\boxed{\arctan x + \arctan\left(\frac{1}{x}\right) = \begin{cases} \frac{\pi}{2} & \text{if } x > 0 \\ -\frac{\pi}{2} & \text{if } x < 0 \end{cases}} \quad (0.5point).$$

• **2$^{\text{nd}}$ way :** Put : $\forall x \in \mathbb{R}^*$, $\arctan x = y$, so,

$$\left\{\begin{array}{l} x = \tan y \\ y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\end{array}\right. \qquad (1point).$$

Yields:

$$\left\{\begin{array}{ll} y \in]0, \frac{\pi}{2}[& \text{if } x > 0 \\ y \in]-\frac{\pi}{2}, 0[& \text{if } x < 0 \end{array}\right. \qquad (1point).$$

First case if $x > 0$, $y \in]0, \frac{\pi}{2}[$ then $\left(\frac{\pi}{2} - y\right) \in]0, \frac{\pi}{2}[$.
Let us recall the known trigonometric relationship:

$$\forall \alpha \neq k\pi \ / \ (k \in \mathbb{Z}), \quad \tan(\frac{\pi}{2} - \alpha) = \frac{1}{\tan \alpha}$$

$$\begin{aligned}
&\Rightarrow \tan\left(\frac{\pi}{2} - y\right) = \frac{1}{\tan y} \text{ (because } y \in]0, \frac{\pi}{2}[\text{)} \\
&\Rightarrow \arctan\left(\tan\left(\frac{\pi}{2} - y\right)\right) = \arctan\left(\frac{1}{\tan y}\right) \\
&\Rightarrow \left(\frac{\pi}{2} - y\right) = \arctan\left(\frac{1}{\tan y}\right) \\
&\Rightarrow \forall x > 0, \ \frac{\pi}{2} - \arctan x = \arctan\left(\frac{1}{\tan(\arctan(x))}\right) \\
&\Rightarrow \forall x > 0, \ \frac{\pi}{2} = \arctan\left(\frac{1}{x}\right) + \arctan x. \quad (1,5points).
\end{aligned}$$

Second case if $x < 0$, then $(-x) > 0$ and since arctan is odd $\quad (0.5point)$. we will have (from the first case):

$$\begin{aligned}
\forall x \quad < \quad 0, \ \frac{\pi}{2} &= \arctan\left(\frac{1}{(-x)}\right) + \arctan(-x) \\
\Rightarrow \frac{\pi}{2} &= -\arctan\left(\frac{1}{x}\right) - \arctan x \quad (1point).
\end{aligned}$$

Yields:

$$\forall x < 0, \ -\frac{\pi}{2} = \arctan\left(\frac{1}{x}\right) + \arctan x$$

Finally,

$$\boxed{\arctan x + \arctan\left(\frac{1}{x}\right) = \left\{\begin{array}{ll} \frac{\pi}{2} & \text{si } x > 0 \\ -\frac{\pi}{2} & \text{si } x < 0 \end{array}\right.}$$

Test 1 Correction: Elementary functions, G1

1. Calculate: $\arcsin\left(\sin\frac{\pi}{6}\right)$.

   Since $\frac{\pi}{6}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ then $\arcsin\left(\sin\frac{\pi}{6}\right)=\frac{\pi}{6}$. $(1point)$.

2. Show that $\forall\alpha\geq 1,\ \arg ch(\alpha)=\ln(\alpha+\sqrt{\alpha^2-1})$.

- 1^{st} method: Put $\arg ch(\alpha)=y$ with $y\geq 0$.

1. If $\arg ch(\alpha)=y$, then $\alpha=chy\Rightarrow\alpha=\dfrac{e^y+e^{-y}}{2}\Rightarrow e^y+e^{-y}-2\alpha=0...(*)$. $(1point)$.

   Put $e^y=z\geq 1$ (because $y\geq 0$), we then find: $e^{-y}=\frac{1}{z}\,(>0)$. So,

$$\begin{aligned}(*)\quad&\Leftrightarrow\quad z+\frac{1}{z}-2\alpha=0\\ &\Leftrightarrow\quad z^2-2\alpha z+1=0\\ &\Leftrightarrow\quad z_{1,2}=\alpha\pm\sqrt{\alpha^2-1}\quad(1point).\end{aligned}$$

- – We have to show that, $z_1=\alpha-\sqrt{\alpha^2-1}<1$, Suppose that :

$$\begin{aligned}\alpha-1-\sqrt{\alpha^2-1}&<0\\ &\Rightarrow\ \alpha-1<\sqrt{\alpha^2-1}\end{aligned}$$

Since $\alpha\geq 1$ then $\alpha-1\geq 0$, so we can apply the square. We find:

$$\begin{aligned}(\alpha-1)^2&<\alpha^2-1\\ -2(\alpha-1)&<0\text{ which is true }\forall\alpha\geq 1.\quad(1point).\end{aligned}$$

z_1 is hence refused.

- – We then show that, $z_2=\alpha+\sqrt{\alpha^2-1}\geq 1$,

Suppose the opposite, i.e. that:

$$\begin{aligned}\alpha-1+\sqrt{\alpha^2-1}&<0\\ &\Rightarrow\ \alpha-1<-\sqrt{\alpha^2-1}\end{aligned}$$

Which is impossible, given that $\alpha-1\geq 0$. Hence, $\forall\alpha\geq 1,\alpha+\sqrt{\alpha^2-1}\geq 1$ $(1point)$.
z_2 is therefore accepted.
Consequently, the only solution is : $z_2=\alpha+\sqrt{\alpha^2-1}\geq 1$.
This gives : $e^y=\alpha+\sqrt{\alpha^2-1}\Rightarrow y=\ln\left(\alpha+\sqrt{\alpha^2-1}\right)$. Hence, $\forall\alpha\geq 1,\ \arg ch(\alpha)=\ln\left(\alpha+\sqrt{\alpha^2-1}\right)$. $(1point)$.

- 2^{nd} method: Put $\alpha=chy$ with $y\geq 0,\alpha\geq 1$ $(1point)$.

We know that $chy+shy=e^y$, so, $y=\ln(chy+shy)$. $(1point)$.
But since, $ch^2y-sh^2y=1$, we get : $shy=\pm\sqrt{ch^2y-1}$. $(1point)$.
sh being increasing, $y\geq 0\Rightarrow shy\geq 0$, which means that $shy=\sqrt{ch^2y-1}$. $(1point)$.
We then obtain $y=\ln(chy+\sqrt{ch^2y-1})$.
And since $y=\arg ch(\alpha)$, we get $\forall\alpha\geq 1,\ \arg ch(\alpha)=\ln\left(\alpha+\sqrt{\alpha^2-1}\right)$. $(1point)$.

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Faculty of Sciences, Mathematics departement M-MI,

Test 1 Correction: Elementary functions, G4 et G5

1. Calculate : $\arccos\left(\cos\frac{2\pi}{3}\right)$.

   Since $\frac{2\pi}{3} \in [0,\pi]$ then $\arccos\left(\cos\frac{2\pi}{3}\right) = \frac{2\pi}{3}$. $(1point)$.

2. Show that $\forall\alpha \in \mathbb{R},\ \arg sh(\alpha) = \ln(\alpha + \sqrt{\alpha^2+1})$.

- 1^{st} method:

We put $\arg sh(\alpha) = y$. So

$$\begin{aligned} \alpha &= shy \Rightarrow \alpha = \frac{e^y - e^{-y}}{2} \\ &\Rightarrow e^y - e^{-y} - 2\alpha = 0...(*). \quad (1point). \end{aligned}$$

Put $e^y = z\ (>0)$, to get : $e^{-y} = \frac{1}{z}\,(>0)$. So,

$$\begin{aligned} (*) &\Leftrightarrow z - \frac{1}{z} - 2\alpha = 0 \\ &\Leftrightarrow z^2 - 2\alpha z - 1 = 0 \\ &\Leftrightarrow z_{1,2} = \alpha \pm \sqrt{\alpha^2+1} \quad (1point). \end{aligned}$$

We want to show that, $\forall\alpha \in \mathbb{R}, \begin{cases} z_1 = \alpha - \sqrt{\alpha^2+1} < 0...(1) \\ z_2 = \alpha + \sqrt{\alpha^2+1} > 0...(2) \end{cases}$,

We know that :

$$\begin{aligned} \forall\alpha &\in \mathbb{R},\ \alpha^2 < \left(\alpha^2+1\right) \\ &\Rightarrow |\alpha| < \sqrt{\alpha^2+1} \\ &\Rightarrow -\sqrt{\alpha^2+1} < \alpha < \sqrt{\alpha^2+1}...(*) \end{aligned}$$

-) $(*) \Rightarrow \forall\alpha \in \mathbb{R},\ \alpha - \sqrt{\alpha^2+1} < 0$. z_1 is then refused. $(1point)$.

-) $(*) \Rightarrow \forall\alpha \in \mathbb{R},\ \alpha + \sqrt{\alpha^2+1} > 0$. z_2 is then accepted. $(1point)$.
The solution then is : $z_2 = \alpha + \sqrt{\alpha^2+1} > 0$.
Which gives that : $e^y = \alpha + \sqrt{\alpha^2+1} \Rightarrow y = \ln\left(\alpha + \sqrt{\alpha^2+1}\right)$. $(1point)$.
Hence, $\forall\alpha \in \mathbb{R},\ \arg sh(\alpha) = \ln\left(\alpha + \sqrt{\alpha^2+1}\right)$.

- **2^{nd}** way

Put $\forall\alpha \in \mathbb{R}, \arg sh(\alpha) = y \Longrightarrow \alpha = shy,\ y \in \mathbb{R}$. $(1point)$.
We know that $\forall y \in \mathbb{R}, chy + shy = e^y$. (Notice that this means that $chy + shy > 0$).
So, $y = \ln(chy + shy)$. $(1point)$.
But since $chy = \sqrt{sh^2y+1}$, $(1point)$. we get : $y = \ln(shy + \sqrt{sh^2y+1})$. $(1point)$.
i.e. $\forall\alpha \in \mathbb{R},\ \arg sh(\alpha) = \ln\left(\alpha + \sqrt{\alpha^2+1}\right)$. $(1point)$.

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Faculty of Sciences, Mathematics departement M-MI,

First and last name :

Test 2 : Derivability and LE, G2

1. Consider the function : **(0.25 pt/right answer)**.

$$f: \quad [0,1[\to [1,+\infty[\qquad x \mapsto f(x) = \frac{1}{\sqrt{1-x^2}}$$

☐ f is **odd**. ☐ f is **even**. ☐ f is **continuous over** $\mathbb{R}$.
☐ f is **continuous over** $[0,1[$. ☐ f is **continuous over** $]0,1[$. ☐ f is **continuous at 1**.
☐ f is **continuous at the right of 1**. ☐ f is **continuous at the right of 0**.
☐ f is **derivable over** $[0,1]$. ☐ f is **derivable over** $[0,1[$. ☐ f is **derivable at 1.**
☐ f is **derivable at the right of 1**. ☐ f is derivable **at the right of 0**. ☐ f admits a **continuous expansion** at 1.

2. Find the $LE_3(0)$ of the function $f(x) = \ln\left(\frac{\sin x}{x}\right)$, then deduce $A = \lim_{x\to 0} \ln\left(\frac{\sin x}{x}\right)$. (6 − (**0.25pt/right answer** in question 1))!

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Faculty of Sciences, Mathematics departement M-MI,

First and last name :

Test 2 : Derivability and LE, G3 and G6

1. Consider the function : **(0.25 pt/right answer)**.

$$f: \quad [0,1[\to [e,+\infty[\\ x \mapsto f(x) = \frac{e}{\sqrt{1-x^2}}$$

☐ f is **odd**. ☐ f is **even**. ☐ f is **continuous over** $\mathbb{R}$.
☐ f is **continuous over** $[0,1[$. ☐ f is **continuous over** $]0,1[$. ☐ f is **continuous at 1**.
☐ f is **continuous at the right of 1**. ☐ f is **continuous at the right of 0**.
☐ f is **derivable over** $[0,1]$. ☐ f is **derivable over** $[0,1[$. ☐ f is **derivable at 1.**
☐ f is **derivable at the right of 1**. ☐ f is derivable **at the right of 0**. ☐ f admits a **continuous expansion** at 1.

2. Find the $\text{LE}_3(0)$ of the function $f(x) = \ln\left(\frac{sh\ x}{x}\right)$, then deduce $A = \lim_{x\to 0} \ln\left(\frac{sh\ x}{x}\right)$. (6 − (**0.25pt/right answer** in question 1))!

First and last name :

Test 2 : Derivability and LE, G1

1. Consider the function : **(1 pt/right answer)**.

$$f: \quad [0,1[\to [\pi,+\infty[\qquad x \mapsto f(x) = \frac{\pi}{\sqrt{1-x^2}}$$

☐ f is **odd**. ☐ f is **even**. ☐ f is **continuous over** $\mathbb{R}$.
☐ f is **continuous over** $[0,1[$. ☐ f is **continuous over** $]0,1[$. ☐ f is **continuous at 1**.
☐ f is **continuous at the right of 1**. ☐ f is **continuous at the right of 0**.
☐ f is **derivable over** $[0,1]$. ☐ f is **derivable over** $[0,1[$. ☐ f is **derivable at 1.**
☐ f is **derivable at the right of 1**. ☐ f is derivable **at the right of 0**. ☐ f admits a **continuous expansion** at 1.

2. Find the LE$_3(0)$ of the function $f(x) = \ln\left(\frac{th\ x}{x}\right)$, then deduce $A = \lim_{x\to 0} \ln\left(\frac{th\ x}{x}\right)$. (6 − (**0.25pt/right answer** in question 1))!

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Faculty of Sciences, Mathematics departement M-MI,

First and last name :

Test 2 : Derivability and LE, G4 and G5

1. Consider the function : **(1 pt/right answer)**.

$$f: \quad [0,1[\to [\sqrt{2},+\infty[$$
$$x \mapsto f(x) = \frac{\sqrt{2}}{\sqrt{1-x^2}}$$

☐ f is **odd**. ☐ f is **even**. ☐ f is **continuous over** $\mathbb{R}$.
☐ f is **continuous over** $[0,1[$. ☐ f is **continuous over** $]0,1[$. ☐ f is **continuous at 1**.
☐ f is **continuous at the right of 1**. ☐ f is **continuous at the right of 0**.
☐ f is **derivable over** $[0,1]$. ☐ f is **derivable over** $[0,1[$. ☐ f is **derivable at 1.**
☐ f is **derivable at the right of 1**. ☐ f is derivable **at the right of 0**. ☐ f admits a **continuous expansion** at 1.

2. Find the $LE_3(0)$ of the function $f(x) = \ln\left(\frac{\tan x}{x}\right)$, then deduce $A = \lim_{x\to 0} \ln\left(\frac{\tan x}{x}\right)$. (6 − (**0.25pt/right answer** in question 1))!

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Test 2 correction : Derivability and LE, G2

1. Consider the function :**(0.25 point/right answer=1.25 points)**.

$$\begin{aligned} f: \quad & [0,1[\to [1,+\infty[\\ & x \mapsto f(x) = \frac{1}{\sqrt{1-x^2}} \end{aligned}$$

☐ f is **odd**. ☐ f is **even**. ☐ f is **continuous over** $\mathbb{R}$.
■ f is **continuous over** $[0,1[$. ■ f is **continuous over** $]0,1[$. ☐ f is **continuous at 1**.
☐ f is **continuous at the right of 1**. ■ f is **continuous at the right of 0**.
☐ f is **derivable at 1**. ■ f is **derivable over** $[0,1[$. ☐ f is **derivable over** $[0,1]$**.**
☐ f is **derivable at the right of 1**. ■ f is **derivable at the right of 0**. ☐ f admits a **continuous expansion** at 1.

2. Find the $LE_3(0)$ of the function $f(x) = \ln\left(\frac{\sin x}{x}\right)$, then deduce $A = \lim_{x\to 0} \ln\left(\frac{\sin x}{x}\right)$. (4.75 *points*).

- Since we will devide by x, we shall consider the LE(0) of $\sin x$ at the order 4 :

$$\sin x = x - \frac{x^3}{6} + o(x^4) \qquad \text{(1 point)}$$

So,

$$\frac{\sin x}{x} = 1 - \frac{x^2}{6} + o(x^3). \quad \text{(0.25 point).}$$

From the other hand, one has the $LE_3(0)$ of the function $\ln(1+t)$ given by :

$$\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} + o(t^3). \quad \text{(1 point).}$$

Put $t = \frac{-x^2}{6}$, so $\lim_{x\to 0} t = 0$. (0.25 point+0.25 point). Then:

$$\begin{aligned} \ln\left(\frac{\sin x}{x}\right) &= \left(-\frac{x^2}{6}\right) - \frac{\left(-\frac{x^2}{6}\right)^2}{2} + \frac{\left(-\frac{x^2}{6}\right)^3}{3} + o\left(\left(-\frac{x^2}{6}\right)^3\right) \\ \Rightarrow \ln\left(\frac{\sin x}{x}\right) &= -\frac{x^2}{6} + o(x^3). \quad \text{(1 point).} \end{aligned}$$

- This means that $A = \lim_{x\to 0} \ln\left(\frac{\sin x}{x}\right) = \lim_{x\to 0}\left[-\frac{x^2}{6} + o(x^3)\right] = 0.$ (1 point).

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Test 2 correction: Derivability and LE, G3 and G6

1. Consider the function :**(0.25 point/right answer=1.25 points)**.

$$f: \quad [0,1[\to [e,+\infty[$$
$$x \mapsto f(x) = \frac{e}{\sqrt{1-x^2}}$$

☐ f is **odd**. ☐ f is **even**. ☐ f is **continuous over** $\mathbb{R}$.
■ f is **continuous over** $[0,1[$. ■ f is **continuous over** $]0,1[$. ☐ f is **continuous at 1**.
☐ f is **continuous at the right of 1**. ■ f is **continuous at the right of 0**.
☐ f is **derivable at 1**. ■ f is **derivable over** $[0,1[$. ☐ f is **derivable over** $[0,1]$**.**
☐ f is **derivable at the right of 1**. ■ f is **derivable at the right of 0**. ☐ f admits a **continuous expansion** at 1.

2. Find the $LE_3(0)$ of the function $f(x) = \ln\left(\frac{sh\ x}{x}\right)$, then deduce $A = \lim_{x\to 0} \ln\left(\frac{sh\ x}{x}\right)$. (4.75 *points*).

- Since we will devide by x, we shall consider the LE(0) of $sh\ x$ at the order 4 :

$$sh\ x = x + \frac{x^3}{6} + o(x^4) \qquad \text{(1 point)}$$

So,

$$\frac{sh\ x}{x} = 1 + \frac{x^2}{6} + o(x^3). \quad \text{(0.25 point).}$$

From the other hand, one has the $LE_3(0)$ of the function $\ln(1+t)$ given by :

$$\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} + o(t^3). \quad \text{(1 point).}$$

Put $t = \frac{x^2}{6}$, so $\lim_{x\to 0} t = 0$. (0.25 point+0.25 point). Then:

$$\ln\left(\frac{sh\ x}{x}\right) = \left(\frac{x^2}{6}\right) - \frac{\left(\frac{x^2}{6}\right)^2}{2} + \frac{\left(\frac{x^2}{6}\right)^3}{3} + o(\left(\frac{x^2}{6}\right)^3)$$
$$\Rightarrow \ln\left(\frac{sh\ x}{x}\right) = \frac{x^2}{6} + o(x^3). \quad \text{(1 point).}$$

- This means that $A = \lim_{x\to 0} \ln\left(\frac{sh\ x}{x}\right) = \lim_{x\to 0}\left[\frac{x^2}{6} + o(x^3)\right] = 0.$ (1 point).

Test 2 Correction: Derivability and LE, G1

1. Consider the function :**(0.25 point/right answer=1.25 points)**.

$$f: \quad [0,1[\to [\pi, +\infty[\qquad x \mapsto f(x) = \frac{\pi}{\sqrt{1-x^2}}$$

☐ f is **odd**. ☐ f is **even**. ☐ f is **continuous over** $\mathbb{R}$.
■ f is **continuous over** $[0,1[$. ■ f is **continuous over** $]0,1[$. ☐ f is **continuous at 1**.
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☐ f is **derivable at 1**. ■ f is **derivable over** $[0,1[$. ☐ f is **derivable over** $[0,1]$.
☐ f is **derivable at the right of 1**. ■ f is **derivable at the right of 0**. ☐ f admits a **continuous expansion** at 1.

2. Find the $LE_3(0)$ of the function $f(x) = \ln\left(\frac{th\ x}{x}\right)$, then deduce $A = \lim_{x\to 0} \ln\left(\frac{th\ x}{x}\right)$. (4.75 *points*).

- Since we will devide by x, we shall consider the LE(0) of $th\ x$ at the order 4 :

$$th\ x = x - \frac{x^3}{3} + o(x^4) \qquad \text{(1 point)}$$

So,

$$\frac{th\ x}{x} = 1 - \frac{x^2}{3} + o(x^3). \quad \text{(0.25 point)}.$$

From the other hand, one has the $LE_3(0)$ of the function $\ln(1+t)$ given by :

$$\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} + o(t^3). \quad \text{(1 point)}.$$

Put $t = \frac{-x^2}{3}$, so $\lim_{x\to 0} t = 0$. (0.25 point+0.25 point). Then:

$$\ln\left(\frac{th\ x}{x}\right) = \left(-\frac{x^2}{3}\right) - \frac{\left(-\frac{x^2}{3}\right)^2}{2} + \frac{\left(-\frac{x^2}{3}\right)^3}{3} + o(\left(-\frac{x^2}{3}\right)^3)$$

$$\Rightarrow \quad \ln\left(\frac{th\ x}{x}\right) = -\frac{x^2}{3} + o(x^3). \quad \text{(1 point)}.$$

- This means that $A = \lim_{x\to 0} \ln\left(\frac{th\ x}{x}\right) = \lim_{x\to 0}\left[\frac{-x^2}{3} + o(x^3)\right] = 0.$ (1 point).

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Test 2 Correction: Derivability and LE, G4 et G5

1. Consider the function :**(0.25 point/right answer=1.25 points)**.

$$f: \quad [0,1[\to [\sqrt{2},+\infty[$$
$$x \mapsto f(x) = \frac{\sqrt{2}}{\sqrt{1-x^2}}$$

☐ f is **odd**. ☐ f is **even**. ☐ f is **continuous over** $\mathbb{R}$.
■ f is **continuous over** $[0,1[$. ■ f is **continuous over** $]0,1[$. ☐ f is **continuous at 1**.
☐ f is **continuous at the right of 1**. ■ f is **continuous at the right of 0**.
☐ f is **derivable at 1**. ■ f is **derivable over** $[0,1[$. ☐ f is **derivable over** $[0,1]$**.**
☐ f is **derivable at the right of 1**. ■ f is **derivable at the right of 0**. ☐ f admits a **continuous expansion** at 1.

2. Find the $LE_3(0)$ of the function $f(x) = \ln\left(\frac{\tan\ x}{x}\right)$, then deduce $A = \lim_{x\to 0} \ln\left(\frac{\tan\ x}{x}\right)$. (4.75 *points*).

- Since we will devide by x, we shall consider the LE(0) of $\tan x$ at the order 4 :

$$\tan x = x + \frac{x^3}{3} + o(x^4) \qquad \text{(1 point)}$$

So,

$$\frac{\tan x}{x} = 1 + \frac{x^2}{3} + o(x^3). \quad \text{(0.25 point).}$$

From the other hand, one has the $LE_3(0)$ of the function $\ln(1+t)$ given by :

$$\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} + o(t^3). \quad \text{(1 point).}$$

Put $t = \frac{x^2}{3}$, so $\lim_{x\to 0} t = 0$. (0.25 point+0.25 point). Then:

$$\ln\left(\frac{\tan x}{x}\right) = \left(\frac{x^2}{3}\right) - \frac{\left(\frac{x^2}{3}\right)^2}{2} + \frac{\left(\frac{x^2}{3}\right)^3}{3} + o(\left(\frac{x^2}{3}\right)^3)$$
$$\Rightarrow \ln\left(\frac{\tan\ x}{x}\right) = \frac{x^2}{3} + o(x^3). \quad \text{(1 point).}$$

- This means that $A = \lim_{x\to 0} \ln\left(\frac{\tan x}{x}\right) = \lim_{x\to 0}\left[\frac{x^2}{3} + o(x^3)\right] = 0.$ (1 point).

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1st grade LMD (2023/2024).
M-MI, 15min.

Test 3 : Integration, G2

1. Integrate $I = \int_{-1}^{1} \frac{\arctan x}{1+x^2}\ dx.$ (1 *point*).
2. Integrate $J = \int_{0}^{1} \frac{\arctan x}{1+x^2}\ dx.$ (2 *points*).
3. Integrate by parts $K = \int \frac{x \arcsin x}{\sqrt{1-x^2}}\ dx.$ (3 *points*).

Test 3 : Integration, G1

1. Integrate $I = \int_{\frac{-1}{2}}^{\frac{1}{2}} \frac{(\arcsin x)^3}{\sqrt{1-x^2}}\ dx.$ (1 *point*).
2. Integrate $J = \int_{0}^{1} \frac{(\arcsin x)^3}{\sqrt{1-x^2}}\ dx.$ (2 *points*).
3. Integrate by parts $K = \int x \arctan x\ dx.$ (3 *points*).

Test 3 : Integration, G3 and G6

1. Integrate $I = \int_{-1}^{1} \frac{(\arctan x)^3}{1+x^2}\ dx.$ (1 *point*).
2. Integrate $J = \int_{0}^{1} \frac{(\arctan x)^3}{1+x^2}\ dx.$ (2 *points*).
3. Integrate by parts $K = \int x(1+\tan^2 x)\ dx.$ (3 *points*).

Test 3 : Integration, G4 and G5

1. Integrate $I = \int_{-1}^{1} \frac{\arcsin x}{\sqrt{1-x^2}}\ dx.$ (1 *point*).
2. Integrate $J = \int_{0}^{1} \frac{\arcsin x}{\sqrt{1-x^2}}\ dx.$ (2 *points*).
3. Integrate by parts $K = \int (x^{n+1}+1) \ln x\ dx.$ $(n \in \mathbb{N})$. (3 *points*).

Test 3 correction : integration, G2

1. Integrate $I = \int_{-1}^{1} \frac{\arctan x}{1+x^2}\ dx$.

   Since $x \to \frac{\arctan x}{1+x^2}$ is an odd function over $[-1,1]$, which is symmetrical with respect to 0, then $I = 0$. (1 *point*).

2. Integrate $J = \int_{0}^{1} \frac{\arctan x}{1+x^2}\ dx$.

   $J = \int_{0}^{1} \frac{\arctan x}{1+x^2}\ dx = \left[\frac{(\arctan x)^2}{2}\right]_0^1 = \frac{(\arctan 1)^2}{2} - \frac{(\arctan 0)^2}{2} = \frac{\left(\frac{\pi}{4}\right)^2}{2} = \frac{\pi^2}{32}$. (2 *points*).

3. Integrate $K = \int \frac{x \arcsin x}{\sqrt{1-x^2}}\ dx$.

   By parts:

$$\text{One takes} \begin{cases} u'(x) = \dfrac{x}{\sqrt{1-x^2}} \\ v(x) = \arcsin x \end{cases} \quad \text{(1 point)},$$

$$\text{to get} \begin{cases} u(x) = \frac{1}{(-2)} \int (-2x)\left(1-x^2\right)^{-\frac{1}{2}} dx = \frac{1}{(-2)} \left[\dfrac{\sqrt{1-x^2}}{\frac{1}{2}}\right] \\ v'(x) = \dfrac{1}{\sqrt{1-x^2}} \end{cases} . \quad \text{(1 point)}$$

The integration by parts formula gives us :

$$\begin{aligned} K &= \int u'v = [uv] - \int v'u \quad \text{(0.5 point)} \\ &= -\arcsin x\sqrt{1-x^2} - \int (-1)\ dx \\ K &= -\arcsin x\sqrt{1-x^2} + x + C, (C \in \mathbb{R}). \quad \text{(0.5 point)} \end{aligned}$$

Test 3 correction: Integration, G1

1. Integrate $I=\int_{\frac{-1}{2}}^{\frac{1}{2}} \frac{(\arcsin x)^3}{\sqrt{1-x^2}}\, dx$.

   Since $x \to \frac{(\arcsin x)^3}{\sqrt{1-x^2}}$ is an odd function over $\left[\frac{-1}{2},\frac{1}{2}\right]$, which is symmetrical with respect to 0, then $I=0$. (1 *point*).

2. Integrate $J=\int_0^1 \frac{(\arcsin x)^3}{\sqrt{1-x^2}}\, dx$.

   $J=\int_0^1 \frac{(\arcsin x)^3}{\sqrt{1-x^2}}\, dx = \left[\frac{(\arcsin x)^4}{4}\right]_0^1 = \frac{(\arcsin 1)^4}{4} - \frac{(\arcsin 0)^2}{4} = \frac{(\frac{\pi}{2})^4}{4} = \frac{\pi^4}{64}$. (2 *points*).

3. Integrate $\int x \arctan x\, dx$.
   By parts:

$$\text{One takes } \begin{cases} u'(x) = x \\ v(x) = \arctan x \end{cases} \quad \text{(1 point)},$$

$$\text{to get } \begin{cases} u(x) = \dfrac{x^2}{2} \\ v'(x) = \dfrac{1}{1+x^2} \end{cases}. \quad \text{(1 point)}$$

The integration by parts formula gives us :

$$\begin{aligned}
K &= \int u'v = [uv] - \int v'u \quad \text{(0.5 point)} \\
&= \frac{x^2}{2}\arctan x - \int \frac{x^2}{2}\frac{1}{1+x^2}\, dx \\
&= \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2+1-1}{1+x^2}\, dx \\
&= \frac{x^2}{2}\arctan x - \frac{1}{2}\int \left(1-\frac{1}{1+x^2}\right) dx \\
K &= \frac{x^2}{2}\arctan x - \frac{1}{2}(x-\arctan x) + C, (C\in\mathbb{R}). \quad \text{(0.5 point)}
\end{aligned}$$

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Test 3 Correction: Integration, G3 and G6

1. Integrate $I = \int_{-1}^{1} \frac{(\arctan x)^3}{1+x^2}\, dx$.

   Since $x \to \frac{(\arctan x)^3}{1+x^2}$ is an odd function over $[-1,1]$, which is symmetrical with respect to 0, then $I = 0$. (1 *point*).

2. Integrate $J = \int_0^1 \frac{(\arctan x)^3}{1+x^2}\, dx$.

   $J = \int_0^1 \frac{\arctan x}{1+x^2}\, dx = \left[\frac{(\arctan x)^4}{4}\right]_0^1 = \frac{(\arctan 1)^4}{4} - \frac{(\arctan 0)^4}{4} = \frac{(\frac{\pi}{4})^4}{4} = \frac{\pi^4}{4^5}$. (2 *points*).

3. Integrate $\int x(1+\tan^2 x)\, dx$.
   By parts:

$$\text{One takes } \begin{cases} u'(x) = 1+\tan^2 x \\ v(x) = x \end{cases} \quad \text{(1 point)},$$
$$\text{to get } \begin{cases} u(x) = \tan x \\ v'(x) = 1 \end{cases} . \quad \text{(1 point)}$$

The integration by parts formula gives us :

$$\begin{aligned} K &= \int u'v = [uv] - \int v'u \quad \text{(0.5 point)} \\ &= x\tan x - \int \tan x dx \\ &= x\tan x - \int \frac{\sin x}{\cos x} dx \\ K &= x\tan x + \ln|\cos x| + C, (C \in \mathbb{R}). \quad \text{(0.5 point)} \end{aligned}$$

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Test 3 Correction: Integration, G4 et G5

1. Integrate $I = \int_{-1}^{1} \frac{\arcsin x}{\sqrt{1-x^2}}\, dx$.

   Since $x \to \frac{\arcsin x}{\sqrt{1-x^2}}$ is an odd function over $[-1, 1]$, which is symmetrical with respect to 0, then $I = 0$. (1 *point*).

2. Integrate $J = \int_{0}^{1} \frac{\arcsin x}{\sqrt{1-x^2}}\, dx$.

   $J = \int_0^1 \frac{\arcsin x}{\sqrt{1-x^2}} dx = \left[\frac{(\arcsin x)^2}{2}\right]_0^1 = \frac{(\arcsin 1)^2}{2} - \frac{(\arcsin 0)^2}{2} = \frac{(\frac{\pi}{2})^2}{2} = \frac{\pi^2}{8}$. (2 *points*).

3. Integrate $\int (x^{n+1} + 1) \ln x \; dx$. $(n \in \mathbb{N})$. (3 *points*).
   By parts:

$$\text{One takes } \begin{cases} u'(x) = (x^{n+1} + 1) \\ v(x) = \ln x \end{cases} \quad \text{(1 point)},$$

$$\text{to get } \begin{cases} u(x) = \int (x^{n+1} + 1) dx = \left(\frac{x^{n+2}}{n+2} + x\right) \\ v'(x) = \dfrac{1}{x} \end{cases} . \quad \text{(1 point)}$$

The integration by parts formula gives us :

$$\begin{aligned} K &= \int u'v = [uv] - \int v'u \quad \text{(0.5 point)} \\ &= \left(\frac{x^{n+2}}{n+2} + x\right) \ln x - \int \left(\frac{x^{n+2}}{n+2} + x\right) \frac{1}{x}\, dx \\ &= \left(\frac{x^{n+2}}{n+2} + x\right) \ln x - \int \left(\frac{x^{n+1}}{n+2} + 1\right) dx \\ K &= \left(\frac{x^{n+2}}{n+2} + x\right) \ln x - \left(\frac{x^{n+2}}{(n+2)^2} + x\right) + C, (C \in \mathbb{R}). \quad \text{(0.5 point)} \end{aligned}$$