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Faculty of Sciences-Department of Mathematics — Module: Algebra 2 L1 Math+MI

Test N° 2

Exercise 1 $(14\ pts)$

Let $E = Span\{(1,1,0),(-3,0,1)\}$, the subspace of $\mathbb{R}^3$ spanned by $B = \{(1,1,0),(-3,0,1)\}$ and $F = \{(x,y,z) \in \mathbb{R}^3 : y + 2x = 0 \text{ and } z - x = 0\}$.

[Soient $E = Vect\{(1,1,0),(-3,0,1)\}$, le s.e.v de $\mathbb{R}^3$ engendré par $B = \{(1,1,0),(-3,0,1)\}$ et $F = \{(x,y,z) \in \mathbb{R}^3 : y + 2x = 0 \text{ et } z - x = 0\}$.]

(1) Show that F is a subspace of $\mathbb{R}^3$. [Montrer que F est un $s.e.v$ de $\mathbb{R}^3$.]

(2) Find a basis for E and a basis for F and deduce $\dim E$ and $\dim F$.

[Trouver une base pour E et une base pour F et en déduire les dimensions de E et F.]

(3) Show that if $v = (x,y,z) \in E$, then $y = x + 3z$.

[Montrer que si $u = (x,y,z) \in E$, alors $y = x + 3z$.]

(4) Show that the subspaces E and F are supplementary in $\mathbb{R}^3$.

[Montrer que les s.e.v E et F sont supplémentaires dans $\mathbb{R}^3$.]

Exercise 2 $(6\ pts)$

(1) Show that $B = \{1, X+1, (X+1)^2\}$ is a basis of $\mathbb{R}_2[X]$.

[Montrer que $B = \{1, X+1, (X+1)^2\}$ est une base de $\mathbb{R}_2[X]$.]

(2) Determine the components of the polynomial $Q(X) = X^2$ in the basis B.

[Déterminer les composantes du polynôme $Q(X) = X^2$ dans la base B.]

Answer

Exercise 1 $(14\ pts)$

(1) Show that F is a subspace of $\mathbb{R}^3$

$$F \text{ is a subspace of } \mathbb{R}^3 \iff \begin{cases} (i) & F \neq \emptyset \quad (0_{\mathbb{R}^3} \in F) \\ (ii) & \forall u,v \in F : u + v \in F \\ (iii) & \forall u \in F, \forall \alpha \in \mathbb{R} : \alpha u \in F \end{cases}$$ (1)

(i) $0_{\mathbb{R}^3} = (0,0,0) \in F$, because $0 + 2.0 = 0$ and $0 - 0 = 0$ then $F \neq \emptyset$. (1)

(ii) Let $u = (x_1,y_1,z_1), v = (x_2,y_2,z_2) \in F \Rightarrow x_1 + 2y_1 = 0,\ z_1 - x_1 = 0$ and $x_2 + 2y_2 = 0,\ z_2 - x_2 = 0$

$u + v = (x_1,y_1,z_1) + (x_2,y_2,z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2) = (X,Y,Z)$

where $X = x_1 + x_2,\ Y = y_1 + y_2,\ Z = z_1 + z_2$

$$\text{We have } \begin{cases} X - 2Y = (x_1 + x_2) + 2(y_1 + y_2) = (x_1 + 2y_1) + (2x_2 + y_2) = 0 + 0 = 0 \\ Z - X = (z_1 + z_2) - (x_1 + x_2) = (z_1 - x_1) + (z_2 - x_2) = 0 + 0 = 0 \end{cases}$$ (2)

So, $u + v \in F$

(iii) Let $u = (x,y,z) \in F$ and $\alpha \in \mathbb{R}$.

$u = (x,y,z) \in F \Rightarrow y + 2x = 0$ and $z - x = 0$

$\alpha u = \alpha(x,y,z) = (\alpha x, \alpha y, \alpha z)$

$$\text{We have } \begin{cases} (\alpha x) + 2(\alpha y) = \alpha(y + 2x) = \alpha.0 = 0 \\ (\alpha z) - (\alpha x) = \alpha(z - x) = \alpha.0 = 0 \end{cases}$$ (2)

Hence $\alpha u \in F$

Conclusion : F is subspace of $\mathbb{R}^3$.

(2) Let's determine $\dim(E)$ and $\dim(F)$.
$E = Span\{(1,1,0),(-3,0,1)\} \Rightarrow B_1 = \{u_1,u_2\}$ is a spanning part of E,

where $u_1 = (1,1,0)$ and $u_2 = (-3,0,1)$.
Is B_1 linearly independent ?
Let $\lambda_1,\lambda_2 \in \mathbb{R}$.
$\lambda_1 u_1 + \lambda_2 u_2 = 0 \Rightarrow \lambda_1(1,1,0) + \lambda_2(-3,0,1) = (\lambda_1 - 3\lambda_2, \lambda_1, \lambda_2) = (0,0,0)$ (2)

$$\Rightarrow \begin{cases} \lambda_1 - 3\lambda_2 = 0 \\ \lambda_1 = 0 \\ \lambda_2 = 0 \end{cases} \Rightarrow \lambda_1 = \lambda_2 = 0$$

Hence $B_1 = \{u_1,u_2\}$ is linearly independent and therefore it is a basis of E,
and since $card(B_1) = 2$ then $\dim(E) = 2$.
Let $u = (x,y,z) \in F \Rightarrow y + 2x = 0$ and $z - x = 0$
$\Rightarrow y = -2x$ and $z = x$
$u = (x,-2x,x) = x(1,-2,1) = xu_3$, where $u_3 = (1,-2,1)$.
We deduce that $F = Span\{u_3\}$, which means : (2)
$B_2 = \{u_3\}$ is a spanning part of F.
Since $u_3 \neq 0_{\mathbb{R}^3}$ then $B_2 = \{u_3\}$ is a basis of F.
And consequently $\dim(F) = 1$.
(3) Let's show that if $v = (x,y,z) \in E$, then $x = y + 3z$.
Asume that $v = (x,y,z) \in E$
$v = (x,y,z) \in E \Rightarrow \exists \alpha,\beta \in \mathbb{R} : v = \alpha u_1 + \beta u_2$
$v = v = \alpha u_1 + \beta u_2 \Rightarrow (x,y,z) = \alpha(1,1,0) + \beta(-3,0,1) = (\alpha - 3\beta, \alpha, \beta)$

$$\Rightarrow \begin{cases} x = \alpha - 3\beta \\ y = \alpha \\ z = \beta \end{cases} \Rightarrow x = y - 3z \Rightarrow y = x + 3z$$ (2)

Hence, if $v = (x,y,z) \in E$, then $y = x + 3z$.
(4) Let's show that the subspaces E and F are supplementary in $\mathbb{R}^3$.
E and F are supplementary in $\mathbb{R}^3 \Leftrightarrow \mathbb{R}^3 = E \oplus F$

$$\mathbb{R}^3 = E \oplus F \Leftrightarrow \begin{cases} \dim(E) + \dim(F) = \dim(\mathbb{R}^3) \\ E \cap F = \{0_{\mathbb{R}^3}\} \end{cases}$$ (1)

We have $\dim(E) + \dim(F) = 2 + 1 = 3 = \dim(\mathbb{R}^3)$
So, to show that $\mathbb{R}^3 = E \oplus F$, it suffices to show that $E \cap F = \{0_{\mathbb{R}^3}\}$.

Let $u = (x,y,z) \in E \cap F \Rightarrow \begin{cases} (x,y,z) \in E \\ (x,y,z) \in F \end{cases}$

$$\Rightarrow \begin{cases} y = x + 3z \\ y + 2x = 0 \\ z - x = 0 \end{cases} \Rightarrow \begin{cases} y - x - 3z + 0 \\ y = -2x \\ z = x \end{cases}$$

$$\Rightarrow \begin{cases} (-2x) - x - 3(x) = 0 \\ y = -2x \\ z = x \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}$$ (1)

Hence $E \cap F = \{0_{\mathbb{R}^3}\}$.

We have $\begin{cases} \dim(E) + \dim(F) = \dim(\mathbb{R}^3) \\ E \cap F = \{0_{\mathbb{R}^3}\} \end{cases} \Leftrightarrow \mathbb{R}^3 = A \oplus B$ (1)

That is E and F are supplementary in $\mathbb{R}^3$.

Exercise 2 $(6\ pts)$

(1) Let's show that $B = \{1, 1+X, (1+X)^2\}$ is a basis of $\mathbb{R}_2[X]$.

As $\dim E = 3$ and $card(B) = 3$, then to show that B is a basis of $\mathbb{R}_2[X]$, it suffices to show that B is linearly independent.

Let Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$.

$$\lambda_1(1) + \lambda_2(1+X) + \lambda_3(1+X)^2 = 0 \Rightarrow \lambda_1 + \lambda_2 + \lambda_2 X + \lambda_3(1 + 2X + X^2) = 0$$
$$= (\lambda_1 + \lambda_2 + \lambda_3) + (\lambda_2 + 2\lambda_3)X + \lambda_3 X^2 = 0$$ (3)

$$\Rightarrow \begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = 0 \\ \lambda_2 + 2\lambda_3 = 0 \\ \lambda_3 = 0 \end{cases} \Rightarrow \begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = 0 \\ \lambda_2 = -2\lambda_3 \\ \lambda_3 = 0 \end{cases} \quad \lambda_1 = \lambda_2 = \lambda_3 = 0$$

Hence $B = \{1, 1+X, (1+X)^2\}$ is linearly independent and therefore it is a basis of $\mathbb{R}_2[X]$.

(2) Let's determine the components of the polynomial $Q(X) = X^2$ in the basis B.

$$Q(X) = X^2 = \alpha(1) + \beta(1+X) + \gamma(1+X)^2 = \alpha + \beta + \beta X + \gamma(1 + 2X + X^2)$$
$$= (\alpha + \beta + \gamma) + (\beta + 2\gamma)X + \gamma X^2$$

$$\Rightarrow \begin{cases} \alpha + \beta + \gamma = 0 \\ \beta + 2\gamma = 0 \\ \gamma = 1 \end{cases} \Rightarrow \begin{cases} \alpha = -\beta - \gamma \\ \beta = -2\gamma \\ \gamma = 1 \end{cases} \Rightarrow \begin{cases} \alpha = 1 \\ \beta = -2 \\ \gamma = 1 \end{cases}$$ (3)

Then $Q(X) = X^2 = \mathbf{1}(1) + (\mathbf{-2})(1+X) + \mathbf{1}.(1+X)^2$