Chapitre 1

Number Systems

In this chapter we will discuss different number systems commonly used to represent data. We begin with the decimal number system and followed by the more commonly used number systems such as the binary, octal and hexadecimal number systems.

Definition 1. *The number system is a mathematical system used to represent and count numbers.*

Definition 2. *Let b be an integer ; $b \geq 2$. Any natural number N can be expressed as a sum of terms of the form $x_i.b^i$, where the numbers i are natural integers, and the numbers x_i are natural integers between 0 and $b-1$. The integers x_i are the digits (or symbols) of the base b.*

For example $(N)_b = x_n.b^n + x_{n-1}.b^{n-1} + \cdots + x_1.b^1 + x_0.b^0$. This decomposition is called the polynomial form of the number N.

1.1 Decimal Number System

The decimal number system is a radix-10 number system and has 10 different digits or symbols. These are $0, 1, 2, 3, 4, 5, 6, 7, 8$ and 9.

Example 1. 6935 can be expressed as : $6935 = 6 \times 10^3 + 9 \times 10^2 + 3 \times 10^1 + 5 \times 10^0$

Remark. For the decimal number system we will not note the indicator 10

1.1.1 Binary Number System

Because digital circuits work with only two voltage states, it is logical to use the binary number system to keep track of information. The binary number system with $'0'$ and $'1'$ as the two independent digits. The procedure for wrinting higher order binary numbers after $'1'$ is similar to the one explained in the case of the decimal number system.

For example ; the first 16 numbers in the binary number system would be

Decimal				Binary
0				0
1				1
2			1	0
3			1	1
4		1	0	0
5		1	0	1
6		1	1	0
7		1	1	1
8	1	0	0	0
9	1	0	0	1
10	1	0	1	0
11	1	0	1	1
12	1	1	0	0
13	1	1	0	1
14	1	1	1	0
15	1	1	1	1

Example 2. A binary number such as 11011_2 (32_{10}) can be expressed as successive powers of 2.

$$11011_2 = 1.2^4 + 1.2^3 + 0.2^2 + 1.2^1 + 1.2^0.$$

1.2 Octal Number and Hexadecimal Number System

Two other number systems used in digital electronics include the octal and hexadecimal systems.

The octal number system has a radix of 8 and therefore has eight distinct digits ; $0, 1, 2, 3, 4, 5, 6$ and 7.

The hexadecimal number system is a radix-16 number system and its 16 basic digits are $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E$ and F ; where $A = 10, B = 11, C = 12, D = 13, E = 14$ and $F = 15$.

Example 3. An octal number such as 2107_8 can be expressed as successive powers of 8.

$$2107_8 = 2.8^3 + 1.8^2 + 0.8^1 + 7.8^0 = 1095.$$

Definition 3. *Bit is an abbreviation of the term 'binary digit' and is the smallest unit of information. It is either '0' or '1'. A byte is a string of eight bits. The byte is the basic unit of data operated upon as a single unit in computers. A computer word is again a string of bits whose size, called the 'word lengh' or 'word size'is fixed for a specified computer. The word lenght may equal one byte, two bytes, four bytes or be even larger.*

1 bite							
1 Byte							

Definition 4. *In a binary number, the bit furthest to the left is called the most significant bit (MSB) and the bit furthest to the right is called the least significant bit (LSB).*

Example 4. $\overset{MSB}{1}\ 100101\ \underset{LSB}{0}$

1.3 Conversion between different systems

1.3.1 Radix-b representation to decimal

For an integer N represented by n digits with radix b, the formula for conversion to decimal representation is as follow :

$$(a_{n-1}a_{n-2}\cdots a_2a_1a_0)_b = \sum_{i=0}^{n-1} a_i b^i = N.$$

Example 5. Convert the binary number 100111_2, the octal number 651_8 and the hexadecimal number $4AC_{16}$ to decimal.

— $100111_2 = 1.2^5 + 0.2^4 + 0.2^3 + 1.2^2 + 1.2^1 + 1.2^0 = 40$,
— $651_8 = 6.8^2 + 5.8^1 + 1.8^0 = 425$,
— $4AC = 4.16^2 + 10.16^1 + 12.16^0 = 1046$.

1.3.2 Decimal to Binary-Octal-Hexadecimal

1. The binary equivalent of decimal number can be found by successively dividing the number by 2 and recording the remainders until the quotient becomes $'0'$. The remainders written in reverse order constitute the binary equivalent.

2. The process of decimal to octal conversion is similar to that of decimal to binary conversion. The division here is by 8.

3. The process of decimal to hexadecimal conversion is also similar. Since the haxadecimal number system has a base of 16, the progressive division in this case is 16.

Remark. The conversion of any decimal number to a number in base b can be hold by successively dividing the bumber by b and recording the remainders until the quotient becomes $'0'$.

The process can be best illustrated with the help of the following example.

69	2					
68	34	2				
1	34	17	2			
	0	16	8	2		
		1	8	4	2	
			0	4	2	2
				0	2	1
					0	

$\longleftarrow$

$69 = 1000101_2$

265	8	
264	33	8
1	32	4
	1	

$\longleftarrow$

$265 = 411_8$

2359	16	
2352	147	16
7	144	9
	3	

$\longleftarrow$

$2359 = 937_{16}$

637	9	
630	70	9
7	63	7
	7	

$\longleftarrow$

$637 = 777_9$

1.3.3 Binary-Octal and Octal-Binary conversions

An octal number can be converted into its binary equivalent by replacing each octal digit with its three-bit binary equivalent. We take the three-bit equivalent because the base of the octal number system is 8 and it is the third power of the base of the binary number system, i.e 2.A binary number can be converted into an equivalent octal number by splitting the number into groups of three bits, starting from the binary point on both sides. The 0s can be added to complete the outside groups if needed.

Example 6. Let us find the binary equivalent of $(3764)_8$.

$$\underbrace{3}_{011}\underbrace{7}_{111}\underbrace{6}_{110}\underbrace{4}_{100},$$

so $(3764)_8 = (11111110100)_2$.

Example 7. Let us find the octal equivalent of $(10011000111)_2$.

$$\underbrace{010}_{2}\ \underbrace{011}_{3}\ \underbrace{000}_{0}\ \underbrace{111}_{7},$$

so $(10011000111)_2 = (10011000111)_2$.

1.3.4 Hexadecimal-Binary and Binary-Hexadecimal conversions

A hexadecimal number can be converted into its binary equivalent by replacing each hex digit with its four-bit binary equivalent. We take the four-bit equivalent because the base of the hexadecimal number system is 16 and it is the fourth power of the base of the binary number system. A given binary number can be converted into an equivalent hexadecimal number by splitting digits into groups of four bits, starting from the binary point on both sides. The 0s can be added to complete the outside groups if needed.

Example 8. Let us find the binary equivalent of $(3569)_{16}$.

$$\underbrace{3}_{0011}\underbrace{5}_{0101}\underbrace{6}_{0110}\underbrace{9}_{1001},$$

so $(3569)_{16} = (0011010101101001)_2$.

Example 9. Let us find the hex equivalent of $(101101000110010111)_2$.

$$\underbrace{0010}_{2}\ \underbrace{1101}_{D}\ \underbrace{0001}_{1}\ \underbrace{1001}_{9}\ \underbrace{0111}_{7},$$

so $(101101000110010111)_2 = 2D197_{16}$

1.3.5 Hex-Octal and Octal-Hex conversions

For hexadecimal-octal conversion, the given hex number is firstly converted into its binary equivalent which is further converted into its octal equivalent. An alternative approach is firstly to convert the given hexadecimal number into its decimal equivalent and then convert the decimal number into an equivalent octal number. For octal-hexadecimal conversion, the octal number may first be converted into an equivalent binary number and then the binary number transformed into its hex equivalent. The other option is firstly to convert the given octal number into its decimal equivalent and then convert the decimal number into its hex equivalent.

Example 10. Let us find the octal equivalent of $(54F)_{16}$.

$$\underbrace{5}_{0101}\underbrace{4}_{0100}\underbrace{F}_{1111},$$

then $(54F)_{16} = \underbrace{010}_{2}\underbrace{101}_{5}\underbrace{001}_{1}\underbrace{111}_{7} = (2517)_8$

Example 11. Let us find the hex equivalent of $(472)_8$.

$$\underbrace{4}_{100}\underbrace{7}_{111}\underbrace{2}_{010},$$

then $(472)_8 = \underbrace{0001}_{1}\underbrace{0011}_{3}\underbrace{1010}_{A} = (13A)_{16}$.

1.4 Fractional number representation

Definition 5. — *Fractional number is a number in the form $\frac{p}{q}$, where p and q are natural numbers and q is not equal to zero.*

— *In radix b representation, a fractional number N has the form.*

$$(N)_b = (a_n a_{n-1} \cdots a_1 a_0 a_{-1} a_{-2} \cdots a_{-p})_b.$$

Example 12. $\frac{9}{6} = 1.333\cdots$, $\frac{10}{4} = 2.5$ are fractional numbers.

1.4.1 Fractional number conversion

1. To convert a fractional number $(N)_b = (a_n a_{n-1} \cdots a_0 a_{-1} a_{-2} \cdots a_{-p})_b$; writing in radix b representation to decimal, we use the polynomial formula : $(N)_{10} = a_n.b^n + a_{n-1}.b^{n-1} + \cdots + a_1.b^1 + a_0.b^0 + a_{-1}.b^{-1} + a_{-2}.b^{-2} + \cdots + a_{-p}.b^{-p}$.
2. To convert a fractional number from decimal to radix b representation, we must
   — convert the whole part by successively dividing by b,
   — convert the fractional part by successively multiply by b and retain the digit that becomes an integer.

   Example 13. Convert the fractional number 215.625 to binary representation.

215	2						
214	107	2					
1	106	53	2				
	1	52	26	2			
		1	26	13	2		
			0	12	6	2	
				1	6	3	2
					0	2	1
						1	

←

$215 = 11010111_2$

   after we first multiply 0.625 by 2 : $0.625 \times 2 = 1.25$, the integer digit is placed in the left column of this table

1	0.25
0	0.5
1	1.0

   . Therefore, $215.625 = (11010111.101)_2$

3. To convert a binary number to octal (hexadecimal); we split the digits into groups of three bits (four bits), starting from the decimal point, moving left for the whole part and moving right for the fractional part. The zeros can be added to cmplete the outside groups if needed.

   Example 14. (a) Convert 100001101.11001_2 to octal representation.
   $(100001101.1001)_2 = (100\ 001\ 101.110\ 010)_2 = (415.62)_8$

   (b) Convert $(100011.101)_2$ to hexadecimal representation.
   $(100011.101)_2 = (1000\ 1100.1010)_2 = (8C.A)_{16}$

1.5 Binary arithmetic

Basic arithmetic operations include addition, subtraction, multiplication and division.

1.5.1 Addition

We can write the basic rules of binary addition as follows

$$
\begin{aligned}
&0 + 0 = 0.\\
&0 + 1 = 1.\\
&1 + 0 = 1.\\
&1 + 1 = 0 \text{ with a carry of } '1' \text{ to the next more significant bit.}\\
&1 + 1 = 1 \text{ with a carry of } '1' \text{ to the next more significant bit.}
\end{aligned}
$$

Example 15. Perform the following Binary addition.

1. $17 + 16 = (10001)_2 + (1111)_2$

$$
\begin{array}{cccccccc}
 & & {}^{1}1 & {}^{1}0 & {}^{1}0 & {}^{1}0 & 1 & (17)\\
+ & & & & & & & \\
 & & & 1 & 1 & 1 & 1 & (15)\\
\hline
= & 1 & 0 & 0 & 0 & 0 & 0 & (32)
\end{array}
$$

2. $11 + 7 = (1011)_2 + (111)_2$

$$
\begin{array}{ccccccc}
 & & {}^{1}1 & {}^{1}0 & {}^{1}1 & 1 & (11)\\
+ & & & & & & \\
 & & & 1 & 1 & 1 & (7)\\
\hline
= & 1 & 0 & 0 & 1 & 0 & (18)
\end{array}
$$

1.5.2 Subtraction

The basic principles of binary subtraction include the following

$$
\begin{aligned}
&0 - 0 = 0.\\
&1 - 0 = 1.\\
&1 - 1 = 0.\\
&0 - 1 = 1 \text{ with a borrow of 1 from the next more significant bit.}
\end{aligned}
$$

Example 16. Perform the following Binary subtraction.

$$
\begin{array}{cccccc}
 & 1 & {}^{1}1 & {}^{1}0 & 1 & (13)\\
- & & & & & \\
 & {}_{1} & {}_{1}1 & 1 & 0 & (6)\\
\hline
= & 0 & 1 & 1 & 1 & (7)
\end{array}
$$

1.5.3 Multiplication

The basic rules of binary multiplication are listed as follows.

$$
\begin{aligned}
&0 \times 0 = 0.\\
&0 \times 1 = 0.\\
&1 \times 0 = 0.\\
&1 \times 1 = 1.
\end{aligned}
$$

The method for multiplication of larger-bit binary numbers is similar to what we familiar with in the case of decimal numbers.

Example 17. Perform the following Binary multiplication. $19 \times 5 = (1011)_2 \times (101)_2$

$$
\begin{array}{ccccccc}
 & & & 1 & 0 & 1 & 1 \\
\times & & & & & & \\
 & & & & 1 & 0 & 1 \\
\hline
 & & & 1 & 0 & 1 & 1 \\
+ & & & & & & \\
 & & {}^{1}0 & 0 & 0 & 0 & . \\
+ & & & & & & \\
 & 1 & 0 & 1 & 1 & . & . \\
\hline
= & 1 & 1 & 0 & 1 & 1 & 1
\end{array}
$$

1.5.4 Division

The algorithm for binary division is some what similar to decimal division.
The binary division rules are as folloxs.

$$0 \div 1 = 0.$$
$$1 \div 1 = 1.$$

Example 18. Perform the following Binary division. $13 \div 5 = (1101)_2 \div (101)_2$

$$
\begin{array}{r|l}
11^{1}01 & 101 \\
\cline{2-2}
-1_{1}01_{\downarrow} & \ 10 \\
\cline{1-1}
 & \\
0011 &
\end{array}
$$