Chapiter 3 : Numerical sequences

Exercise 1

Let be (U_n) a number sequence, and (V_n) , (W_n) two subsequences such that :
$V_n = U_{2n}$ and $W_n = U_{2n+1}$

Show that : $(\lim U_n = l) \Leftrightarrow (\lim V_n = l \text{ et } \lim W_n = l)$.

Then deduce that, if : (U_{2n}) , (U_{2n+1}) , (U_{3n}) converge, then : (U_n) converges too.

Exercise 2

Using the definition, show that :

$$\lim \frac{n+1}{2n+1} = \frac{1}{2} \text{ , } \lim \frac{n-1}{n+1} = 1 \text{ , } \lim \sqrt{n^2+n+1} = +\infty$$

Exercise 3

Find the limit of the following sequences when it exists :

$$1)\ U_n = \frac{n^2+1}{2n^3-1} \text{ ,} \quad 2)\ U_n = \frac{an+b}{cn+d} \text{ ,} \quad 3)\ U_n = \sqrt{n} - \sqrt{n+1}$$

$$4)\ U_n = \prod_{k=2}^{n} \left(1 - \frac{1}{k}\right), \quad 5)\ U_n = \sum_{k=0}^{n} \frac{1}{2^k} \text{ ,} \quad 6)\ U_n = \frac{\sin n}{n}$$

$$7)\ U_n = \cos(n) \sin\left(\frac{1}{n}\right) \text{ ,} \quad 8)\ U_n = n \sin\left(\frac{1}{n}\right), \quad 9)\ U_n = \sum_{k=1}^{n} \frac{1}{n+k}$$

$$10)\ U_n = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \text{ ,} \quad 11)\ U_n = \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} \text{ ,} \quad 12)\ U_n = \sum_{k=1}^{n} \frac{n}{n^2+k}$$

$$13)\ U_n = (-1)^n - \frac{1}{n}.$$

Exercise 4

1. Demonstrate that the sequence defined by : $U_n = \sum_{k=1}^{n} \frac{1}{k}$ is not a Cauchy one .

Deduce then that the sequence of the general term : $V_n = \sum_{k=2}^{n} \frac{1}{\ln(k)}$ is not convergent.

2. Show that the sequence defined by : $U_n = \sum_{k=1}^{n} \frac{\sin k}{2^k}$ is a Cauchy one.

3. Same question as in 2. for the following sequence, defined by its general term : $U_n = \sum_{k=1}^{n} \frac{1}{k^2}$

Exercise 5 The Fibonacci sequence

Let the sequence $(U_n)_{n\in\mathbb{N}}$ defined by :

$$\begin{cases} \forall n \in \mathbb{N} : U_{n+2} = U_{n+1} + U_n \\ U_0 = 0 \qquad U_1 = 1 \end{cases}$$

1. Show that : $\forall n \in \mathbb{N} : U_{n+2}U_n - U_{n+1}^2 = (-1)^{n+1}$
2. Verify that $\dfrac{U_{n+1}}{U_n}$ (n>0), might be written in the form : $w_{n+1} = f(w_n)$
3. Show that : $\left(\dfrac{U_{n+1}}{U_n}\right)_{n\in\mathbb{N}^*}$ converges, then find its limit.

Exercise 6

Discuss the following sequences :

1. $U_0 \in \mathbb{R}, U_{n+1} = \dfrac{1}{2}\sqrt{U_n^2 + 12}$
2. $U_0 \in \mathbb{R}_+, U_{n+1} = \dfrac{1}{2}\sqrt{2U_n + 3}$ (Additional)
3. $U_0 \in \mathbb{R}, U_{n+1} = \dfrac{U_n}{U_n^2 + 1}$
4. $U_0 \in \mathbb{R}, U_0 \neq 1, U_{n+1} = \dfrac{U_n^2 + 1}{U_n - 1}$
5. $U_0 = 3, U_{n+1} = 1 + \dfrac{1}{U_n - 1}$
6. $U_0 = \sqrt{2}, U_{n+1} = \sqrt{2 + U_n}$ (Additional)